POLITEKNIK MALAYSIA MERLIMAU

PARTIAL FRACTION

A SELF-LEARNING GUIDE

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\}$$

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A SELF-LEARNING GUIDE

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PREFACE

Over the years, syllabuses of existing courses have undergone some modifications. Partial Fraction: A Self-Learning Guide has been written to match the latest mathematics syllabuses of the diploma programme for the first, second and third semesters. Topics covered include the properties of Partial Fraction, Integration with Partial Fraction Method and Inverse Laplace Transforms by using Partial Fraction. This book provides the repetition of basic partial fraction principle for effective learning.

Partial fraction is the key for the understanding and solving most algebraic problems. This book will give the students a simpler and easier method for presenting the important principles of partial fraction and demonstrating the best solving techniques in answering the exercises. With concise notes and formulas to highlight some important key points, the users are hoped to have a better comprehension and appreciation of the theory learned… A complete set of answers is available at the end of the book. We hope that users will be able to fully utilise this book in assisting them in the process of teaching and learning.

We are grateful for the opportunity to produce this book. We would like to personally thank the publisher, Penerbit PMM and the e-learning teams of PMM for their support.

Please do not hesitate to contact us with any suggestions or comments that could improve the content of this book.

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CHAPTER OUTCOMES

At the end of this chapter, you should be able to

1. Recognize simple linear factors, repeated linear factors and quadratic factors in a rational function.
2. Decompose a rational expression with denominator of linear factors, repeated linear factors and quadratic factor into a sum of partial fractions.

ChapTer 1 : ParTiaL FracTion

1.0 INTRODUCTION

Partial Fractions are a mathematical concept used to simplify complex rational expressions by breaking them down into simpler fractions. When dealing with fractions that have algebraic expressions, it can be challenging to solve them directly. By using partial fractions, we can split these fractions into multiple simpler fractions, making them easier to work with and solve. During the process of decomposing a fraction into partial fractions, the denominator, which is an algebraic expression, is factored in to simplify the generation of the partial fractions.

The concept of partial fractions is the reverse of the process of adding rational expressions. Instead of combining fractions, partial fractions involve breaking down a single fraction into multiple simpler fractions.

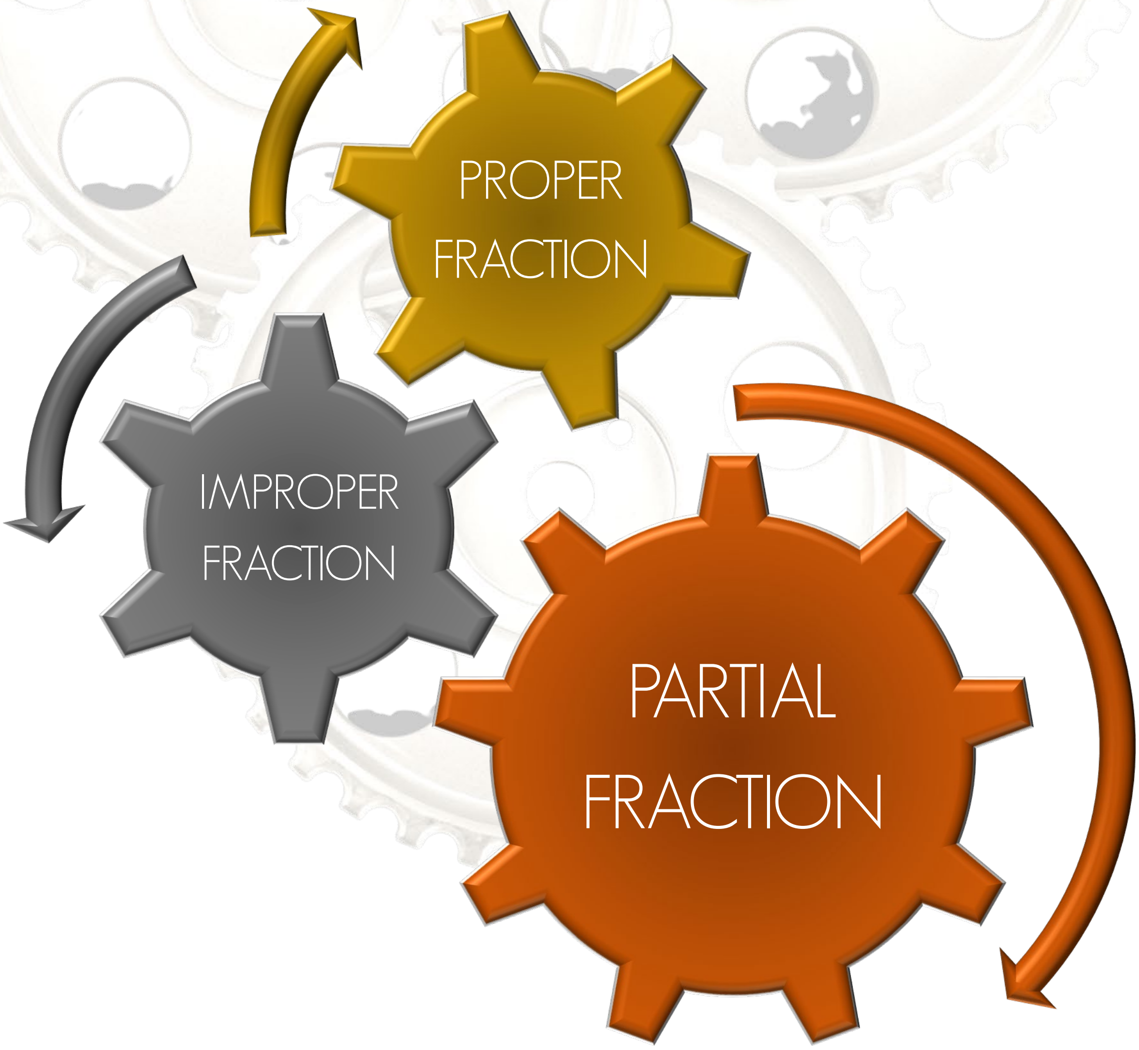
Arithmetic operations across algebraic fractions are performed to obtain a single rational expression. This rational expression, on splitting in the reverse direction involved the process of decomposition of partial fractions and resulted in the two partial fractions.

In partial fractions, the dividend (upper number) is called the numerator $N(x)$ and the divisor (lower number) is called the denominator, $D(x)$.

$$\frac{Numerator\ (x)}{Denominator\ (x)} = \frac{N(x)}{D(x)}$$

1.1 Partial Fraction

It is a return from the single simplified rational expression to the original expressions, called the partial fraction. (Abramson, 2021)



1.1.1 Proper Fraction

Proper fractions, in the context of partial fractions, refer to fractions where the degree of the numerator is less than the degree of the denominator. There are different types of proper fractions, depending on the specific characteristics of the numerator and denominator

a. Non-repeated Linear Factor

The denominator can be factored into distinct linear factors, each factor becomes a separate fraction in the decomposition

$$\frac{N(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \cdots$$

b. Repeated Linear Factor

Partial fraction has repeated factors when one of the denominator factors has multiplicity greater than 1. The process for repeated factors is slightly different than the process for linear, non-repeated factors.

$$\frac{N(x)}{(ax+b)^2} = \frac{A}{(ax+b)^2} + \frac{B}{(ax+b)}$$

c. Quadratic Factor

One of the denominator factors is a quadratic with irrational or complex roots

$$\frac{N(x)}{(ax^2+b)} = \frac{Ax+B}{(ax^2+b)}$$

Partial fractions can also be found in the form of non-repeated linear and quadratic factor combinations. It is called Linear Quadratic Factor

$$\frac{N(x)}{(ax+b)(cx^2+d)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$$

Let's take a look at these set of examples below.

EXAMPLE 1 | NON-REPEATED LINEAR FACTOR

Evaluate the constants A, B and construct the partial fraction for $\frac{x+5}{x^2-3x+2}$

Solution :

Step 1: Factor the denominator of the rational expression.

$$\frac{x+5}{x^2-3x+2} = \frac{x+5}{(x-2)(x-1)}$$

Scan me

Step 2 : Transform to linear factor form : $\frac{A}{(ax+b)} + \frac{B}{(cx+d)}$

$$= \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

Step 3 : Multiply each of the numerator to make the denominator equal..

$$= \frac{A}{(x-2)} \times \frac{(x-1)}{(x-1)} + \frac{B}{(x-1)} \times \frac{(x-2)}{(x-2)}$$

$$= \frac{A(x-1) + B(x-2)}{(x-2)(x-1)}$$

Step 4 : Now find the value of A and B

Method 1 | Elimination

$$x + 5 = A(x-1) + B(x-2)$$

If $x - 1 = 0, x = 1$

$$1 + 5 = A(1-1) + B(1-2)$$

$$6 = -B$$

$$\boldsymbol{B = -6}$$

If $x - 2 = 0, x = 2$

$$2 + 5 = A(2-1) + B(2-2)$$

$$\boldsymbol{7 = A}$$

Method 2 | Expand, equate the coefficient and solve by using calculator.

$$x + 5 = A(x - 1) + B(x - 2)$$

Once Expanded

$$x + 5 = Ax - A + Bx - 2B$$

By equating the coefficients of x and C (constant)

x	C
$A + B = 1$	$-A - 2B = 5$

$$1A + 1B = 1 \quad (a_1, b_1, c_1)$$

$$-1A - 2B = 5 \quad (a_2, b_2, c_2)$$

The values A and B by using a Calculator.

$\boldsymbol{A = 7}$

$\boldsymbol{B = -6}$

Scan me

Scan this QR code to find value A and B by using calculator.

Answer

$$= \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

$$= \frac{\mathbf{7}}{(\boldsymbol{x} - \mathbf{2})} - \frac{\mathbf{6}}{(\boldsymbol{x} - \mathbf{1})}$$

EXAMPLE 2 | REPEATED LINEAR FACTOR

Evaluate the constants A, B and construct the partial fraction for $\dfrac{x^2+3}{x(x+2)^2}$

Solution :

Step 1: Transform to linear factor form : $\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)^2}+\dfrac{C}{(cx+d)}$

$$=\frac{A}{x}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)}$$

Step 2 : Multiply each of numerator to make the denominator equal.

$$=\frac{A}{x}\times\frac{(x+2)^2}{(x+2)^2}+\frac{B}{(x+2)^2}\times\frac{x}{x}+\frac{C}{(x+2)}\times\frac{x(x+2)}{x(x+2)}$$

$$=\frac{A(x+2)^2+Bx+Cx(x+2)}{(x+2)^2}$$

Step 3 : Now find the value of A, B and C

Method 1 | Elimination

$$x^2+3=A(x+2)^2+Bx+Cx(x+2)$$

If $x+2=0, x=-2,$

$$(-2)^2+3=A(-2+2)^2+B(-2)+C(-2)(-2+2)$$

$$7=-2B$$

$$\boldsymbol{B=-\frac{7}{2}}$$

When $x = 0,$

$$(0)^2 + 3 = A(0+2)^2 + B(0) + C(0)(0+2)$$

$$3 = 4A$$

$$\boldsymbol{A = \frac{3}{4}}$$

Since the value of x in the denominator of the question has been substituted in the equation, we can now choose any number of x to be substituted. For example, we choose $x = 1$ plus, we must substitute the values of A, or B, or C that has been solved before.

$$(1)^2 + 3 = \frac{3}{4}(1+2)^2 + (-\frac{7}{2})(1) + C(1)(1+2)$$

$$4 = \frac{27}{4} - \frac{7}{2} + 3C$$

$$C = \frac{4 - \frac{27}{4} + \frac{7}{2}}{3}$$

$$\boldsymbol{C = \frac{1}{4}}$$

Method 2 | Expand, equate the coefficient and solve by using calculator.

$$x^2 + 3 = A(x+2)(x+2) + Bx + Cx(x+2)$$

$$x^2 + 3 = Ax^2 + 4Ax + 4A + Bx + Cx^2 + 2Cx$$

once expanded

By equating the coefficients of x^2, x and C (constant)

x^2	x	C
$A + C = 1$	$4A + B + 2C = 0$	$4A = 3$

The values A, B and C by using a Calculator.

$$A = \frac{3}{4}$$

$$B = -\frac{7}{2}$$

$$C = \frac{1}{4}$$

Scan me

Scan this QR code to find the values A, B and C by using calculator.

REFER TO PAGE 8 to find values $a_1, b_1, c_1, d_1 \ldots\ldots d_3$

Answer

Thus, $\dfrac{x^2+3}{x(x+2)^2} = \dfrac{3}{4x} - \dfrac{7}{2(x+2)^2} + \dfrac{1}{4(x+2)}$

EXAMPLE 3 | LINEAR QUADRATIC FACTOR

1. Evaluate the constants A, B, C and construct the partial fraction for $\frac{3x^2 - 4}{x(x^2+1)}$

Solution :

Step 1: Transform to quadratic factor form : $\frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$

$$\frac{3x^2 - 4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{(x^2+1)}$$

Step 2 : Multiply each of numerator to make the denominator equal.

$$= \frac{A}{x} \times \frac{(x^2+1)}{(x^2+1)} + \frac{Bx + C}{(x^2+1)} \times \frac{x}{x}$$

$$= \frac{A(x^2+1) + (Bx + C)x}{x(x^2+1)}$$

Step 3 : Now find the value of A, B and C.

Method 1 | Elimination and Simultaneous Equation

$$3x^2 + 4 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$,

$$3(0)^2 + 4 = A(0^2 + 1) + (B(0) + C)0$$

$$\mathbf{4 = A}$$

Since the value of x in the denominator of the question has been substituted in the equation, we can now choose any number of x to be susbtituted. For example, we choose $x = 1$ plus, we must substitute the values of A, or B, or C that has been solved before.

$$3(1)^2 + 4 = 4((1)^2 + 1) + (B(1) + C)1$$

$$7 = 8 + B + C$$

$$\boldsymbol{B + C = -1}$$

Equation 1

Use any value of x once again except the value that has been used before. For example $x = 0, x = 1$, so we choose $x = 2$.

$$3(2)^2 + 4 = 4((2)^2 + 1) + (B(2) + C)2$$

$$16 = 20 + 4B + 2C$$

$$4B + 2C = -4$$

$$(4B + 2C = -4)/2$$

$$\boldsymbol{2B + C = -2}$$

Equation 2

Equation 1 - Equation 2

Simultaneous Equation

$$B + C = -1$$

$$(-) \quad 2B + C = -2$$

$$-B = 1$$

$$\boldsymbol{B = -1}$$

Subtitute $B = -1$ in Equation 1

$$B + C = -1$$

$$-1 + C = -1$$

$$\boldsymbol{C = 0}$$

Method 2 | Expand and equating the coefficient.

$$3x^2 + 4 = A(x^2 + 1) + (Bx + C)x$$

$$3x^2 + 4 = Ax^2 + A + Bx^2 + Cx$$

once expanded

By equating the coefficients of x and C (constant)

x^2	x	C
$A + B = 3$	$\boldsymbol{C = 0}$	$\boldsymbol{A = 4}$

So,

$$\boldsymbol{A = 4}$$

$$A + B = 3$$

$$4 + B = 3$$

$$\boldsymbol{B = -1}$$

$$\boldsymbol{C = 0}$$

Answer

That is, $\frac{A}{x} + \frac{Bx+C}{(x^2+1)} = \frac{4}{x} + \frac{(-1)x+0}{(x^2+1)}$

$$= \boldsymbol{\frac{4}{x} - \frac{x}{(x^2+1)}}$$

2. Evaluate the constants A, B, C and construct the partial fraction for

$$\frac{5x^2 + 9x - 4}{(x+1)(x^2+4)}$$

Solution :

Step 1 : We must try, first using calculator if the quadratic equation $x^2 + 4$ can be factorized, if can factorized, we must used linear factor. If cannot we proceed to quadratic factor method.

Step 2 : Transform to linear factor form : $\frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$

$$= \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+4)}$$

Step 3 : Multiply each of numerator to make the denominator equal.

$$= \frac{A}{(x+1)} \times \frac{(x^2+4)}{(x^2+4)} + \frac{Bx+C}{(x^2+4)} \times \frac{(x+1)}{(x+1)}$$

$$= \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

Step 4 : Now find the value of A, B and C.

Method 1 | Elimination and Subtitution

$$5x^2 + 9x - 4 = A(x^2+4) + (Bx+C)(x+1)$$

When $x + 1 = 0, x = -1$

$$5(-1)^2 + 9(-1) - 4$$

$$= A[(-1)^2 + 4] + [B(-1) + C](-1 + 1)$$

$$-8 = 5A$$

$$\boldsymbol{A = \frac{-8}{5}}$$

Since the value of x in the denominator of the question has been substituted in the equation, we can now choose any number of x to be substituted. For example, we choose $\boldsymbol{x = 0}$ plus, we must substitute the values of $\boldsymbol{A}$, or $\boldsymbol{B}$, or $\boldsymbol{C}$ that has been solved before.

$$5(0)^2 + 9(0) - 4 = \frac{-8}{5}[(0)^2 + 4] + [B(0) + C](0 + 1)$$

$$-4 = \frac{-32}{5} + C$$

$$\boldsymbol{C = \frac{12}{5}}$$

Use any value of $\boldsymbol{x}$ once again except the value that has been used before. For example $\boldsymbol{x = 0, x = 1}$, so we choose $\boldsymbol{x = 1}$.

$$5(1)^2 + 9(1) - 4 = \frac{-8}{5}[(1)^2 + 4] + \left[B(1) + \frac{12}{5}\right](1 + 1)$$

$$10 = -8 + 2B + \frac{24}{5}$$

$$2B - \frac{16}{5} = 10$$

$$\boldsymbol{B = \frac{33}{5}}$$

Method 2 | Expand, equate the coefficient and solve by using calculator.

$$5x^2 + 9x - 4 = A(x^2 + 4) + (Bx + C)(x + 1)$$

$$5x^2 + 9x - 4 = Ax^2 + 4A + Bx^2 + Bx + Cx + C$$

once expanded

By equating the coefficients of x and C (constant)

x^2	x	C
$A + B = 5$	$B + C = 9$	$4A + C = -4$

Find the values A, B and C by using a Calculator.

$$\boldsymbol{A = \frac{-8}{5}}$$

$$\boldsymbol{B = \frac{33}{5}}$$

$$\boldsymbol{C = \frac{12}{5}}$$

Scan me

Scan this QR code to find the values A, B and C by using calculator.

REFER TO PAGE 8 to find values $a_1, b_1, c_1, d_1 \dots\dots. d_3$

Answer

Thus, $\frac{5x^2+9x-4}{(x+1)(x^2+4)} = \frac{-8}{5(x+1)} + \frac{33x+12}{5(x^2+4)}$

DRILL YOUR BRAIN 1

1. Express $\frac{8x-42}{(x-3)(x+6)}$ in partial fraction.

Solution :

Step 1: Transform to Non-Repeated linear/Repeated/Quadratic factor form

Step 2 : Multiply each of numerator to make the denominator equal.

Step 3 : Now, find the value of A and B

Method 1 | Elimination

Method 2 | Expand and equate the coefficients.

Answer Thus, $\frac{8x-42}{(x-3)(x+6)} = \frac{-2}{x-3} + \frac{10}{x+6}$

2. Express $\frac{x^2-2x+3}{(x+1)(x^2+5)}$ in partial fraction.

Solution :

Step 1: Factor the denominator of the rational expression

Step 2 : Transform to Non-Repeated linear/Repeated/Quadratic factor form

Step 3 : Make the denominator equal

Step 3 : Now, find the value of A, B and C

Method 1 | Elimination and Simultaneous Equation

Method 2 | Expand and equate the coefficients.

Answer

Thus, $\frac{x^2-2x+3}{(x+1)(x^2+5)} = \frac{1}{(x+1)} - \frac{2}{(x^2+5)}$

3. Express $\frac{5x-3}{(x+2)(x-3)^2}$ in partial fraction.

Solution :

Step 1: Transform to Non-Repeated linear/Repeated/Quadratic factor form

Step 2 : Multiply each of the numerator to make the denominator equal.

Step 3 : Now find the value of A, B and C

Method 1 | Elimination

Method 2 | Expand and equate the coefficients.

Answer Thus, $\frac{5x-3}{(x+2)(x-3)^2} = -\frac{13}{25(x+2)} + \frac{12}{5(x-3)^2} + \frac{13}{25(x-3)}$

1.1.2 Improper Fraction

An improper fraction is a type of fraction where the numerator, or the top number, is equal or larger than the denominator or the bottom number. This means that the fraction represents a value greater than one. Improper fractions are often converted into mixed numbers, which are a combination of a whole number and a proper fraction, to make them easier to work within certain situations. Polynomial long division is needed to reduce the improper expression to a mixed form consisting of the sum of a polynomial and a proper expression

Example :

A polynomial

Proper expression

$$\frac{x^4 + 4}{x^2 - 5} = x^2 + 5 + \frac{29}{x^2 - 5}$$

Let's take a look at these set of examples below.

EXAMPLE 4 | LONG DIVISION OF POLYNOMIALS

Construct the partial fraction for $$\frac{3x^3 - 2x^2 + 4x + 7}{x^2 + 2x}$$

Solution :

Step 1: Arrange term between both numerator and denominator in descending variable degrees.

$$\frac{3x^3 - 2x^2 + 4x + 7}{x^2 + 2x}$$

Step 2 : Divide using long division

$$\begin{array}{r|l} & \boxed{3x - 8} \ ① \\ ③\ \boxed{x^2 + 2x} & \overline{3x^3 - 2x^2 + 4x + 7} \\ (-) & \underline{3x^3 + 6x^2} \\ & -8x^2 + 4x \\ (-) & \underline{-8x^2 - 16x} \\ & \boxed{20x + 7} \ ② \end{array}$$

Step 3 : Thus, write the proper fraction form

$$\underset{①}{3x - 8} + \frac{\overset{②}{20x + 7}}{\underset{③}{x^2 + 2x}}$$

Solve in partial fraction

Step 4 : Factor the denominator of the rational expression.

$$3x - 8 + \frac{20x + 7}{x^2 + 2x}$$

$$= 3x - 8 + \frac{20x + 7}{x(x + 2)}$$

How to factorize the denominator using calculator?
Click here to refer

Page 5

Step 5 : Transform to linear factor form : $\frac{A}{(ax+b)} + \frac{B}{(cx+d)}$

$$\frac{20x + 7}{x^2 + 2x} = \frac{A}{x} + \frac{B}{(x + 2)}$$

Only the fraction $\frac{20x+7}{x(x+2)}$ is implemented partial fraction.

Step 6 : Multiply each of numerator to make the denominator equal.

$$= \frac{A}{x} \times \frac{(x + 2)}{(x + 2)} + \frac{B}{x + 2} \times \frac{x}{x}$$

$$= \frac{A(x + 2) + Bx}{x(x + 2)}$$

Step 7 : Now find the values of A and B

Method 1 | Elimination equation

$$20x + 7 = A(x + 2) + Bx$$

If $x = 0,$

$$20(0) + 7 = A(0 + 2) + B(0)$$

$$7 = 2A$$

$$\boldsymbol{A = \frac{7}{2}}$$

If $x = -2,$

$$20(-2) + 7 = A(-2 + 2) + B(-2)$$

$$-33 = -2B$$

$$\boldsymbol{B = \frac{33}{2}}$$

Method 2 | Expand and equate the Coefficient

$$20x + 7 = A(x + 2) + Bx$$

$$20x + 7 = Ax + 2A + Bx$$

once expanded

By equating the coefficients of x and C (constant)

x	C
$\boldsymbol{20 = A + B}$	$\boldsymbol{2A = 7}$

Find the values A, and B by using a Calculator.

$$A = x = \frac{7}{2}$$

$$B = y = \frac{33}{2}$$

How to find value of A and B by using calculator?

Click here to refer

Page 7

Answer That is,

$$\frac{20x+7}{x(x+2)} = \frac{7/2}{x} + \frac{33/2}{(x+2)}$$

Thus, the corrected partial fractions is

$$\frac{3x^3 - 2x^2 + 4x + 7}{x^2 + 2x} = 3x - 8 + \frac{7}{2x} + \frac{33}{2(x+2)}$$

DRILL YOUR BRAIN 2

Express the fraction in a partial fraction $\dfrac{x^2}{x^2 + x - 2}$

Solution :

Step 1: Arrange term between both numerator and denominator in descending variable degrees.

Step 2 : Divide using long division

Step 3 : Write the proper fraction form

Solve in partial fraction

Step 4 : Factor the denominator of the rational expression.

Step 5 : Transform to partial fraction form (linear/repeated/quadratic):

Step 6 : Multiply each of the numerator to make the denominator equal.

Step 7 : Now find the values of A and B

Method 1 | Elimination equation

Method 2 | Expand and Compare the Coefficient

Answer Thus, the corrected partial fractions is

$$\frac{x^2}{x^2+x-2} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{3(x-1)}} + \frac{\mathbf{4}}{\mathbf{3(x+2)}}$$

PRACTICE YOUR KNOWLEDGE 1

LET'S DO IT

Express the following fractions in a partial fraction.

1. $\dfrac{x}{(x+1)(x+2)}$
2. $\dfrac{x+3}{(x-1)(x-2)}$
3. $\dfrac{2x-4}{-2x^2-x+1}$
4. $\dfrac{x+2}{x^2-3x}$
5. $\dfrac{2x-1}{2x^2-5x-3}$
6. $\dfrac{3x^2-7x+5}{(x+2)(x+1)^2}$
7. $\dfrac{3x}{(x-1)^2}$
8. $\dfrac{2x+3}{x^2(1-x)}$
9. $\dfrac{18x+20}{(3x+4)^2}$
10. $\dfrac{x+1}{x(x-1)^3}$
11. $\dfrac{10+5x-x^2}{(x^2+2)(x+1)}$
12. $\dfrac{8x^2-12}{x(x^2+2x-6)}$
13. $\dfrac{x^2+1}{x(x+1)(x-1)}$
14. $\dfrac{5x+13}{x^2+4x-5}$
15. $\dfrac{x^3-2x^2+3}{x^2+5x+4}$
16. $\dfrac{-x+10}{(x+5)(x-3)}$
17. $\dfrac{x^2}{x^2-5x+6}$
18. $\dfrac{1}{x^3(1-2x)}$
19. $\dfrac{x^2}{(x^2+2x+1)(x-3)}$
20. $\dfrac{(5+3x)^2}{(x-5)^2(3x+1)}$
21. $\dfrac{x^2}{(x+1)(x-3)}$
22. $\dfrac{11-5x}{x^2-8x+16}$
23. $\dfrac{x^5+4}{x^3-2x}$
24. $\dfrac{2(1+x)}{x(x^2+4)}$

25. $\dfrac{x}{(x+2)(x^2+5)}$

26. $\dfrac{3x^2+2x}{(x+2)(x^2+3)}$

27. $\dfrac{x^2+1}{x^2-1}$

28. $\dfrac{x^2-x+1}{x^2-x-2}$

29. $\dfrac{x^3+2x^2-10x-9}{x^2-9}$

30. $\dfrac{2x^2}{(2x-1)(x-3)}$

31. $\dfrac{2x^3-x^2+x+5}{x^2+3x+2}$

32. $\dfrac{4x-1}{x^2(x^2-4)}$

33. $\dfrac{x-3}{x^3+3x}$

34. $\dfrac{x^3}{(x+2)(x-3)}$

35. $\dfrac{x^2}{x^2-5x+6}$

36. $\dfrac{x^2-1}{x-2}$

37. $\dfrac{x^3-2x^2+3}{x^2+5x+4}$

38. $\dfrac{4x^3-3x+2}{(2x-1)(x+2)}$

LET'S CHECK YOUR ANSWER!

SCAN ME!

2

CHAPTER OUTCOMES

At the end of this chapter, you should be able to

Solve Integration of Partial Fraction

1. Solve integrals of proper fractions involving:
   i. Linear Factor
   ii. Repeated Linear Factor
   iii. Quadratic Factor
2. Solve integrals of improper fractions.

ChapTer 2 : InteGratioN of ParTiaL FracTion

2.0 INTRODUCTION

In this section, we will look at how we can integrate some algebraic fractions. We will be using partial fractions to rewrite the integrand as the sum of simpler fractions that can be then integrated separately. Decomposing a fraction into its component parts and then integrating normally is known as integration using partial fractions.

Before learning about the integration process by partial fractions, we must first, understand the meaning of partial fractions and how to write the partial fractions. The sum of the proper rational functions that resulted from decomposing an improper rational function is known as a partial fraction. In this case, an improper rational function is one where the degree of the numerator exceeds the degree of the denominator. To break down improper rational functions and obtain partial fractions, various formulas are available.

The method of partial fractions is used to integrate rational functions. That is, we want to compute

$\int \frac{P(x)}{Q(x)} dx$ where P, Q are polynomials.

1

Why Use It?

It simplifies integrals and reduces them to a series of simpler integrals that can be more easily evaluated.

2

What Will We Learn?

We will learn how to decompose a difficult fraction into simpler fractions and how to compute integrals using these simpler fractions.

2.1 How to do integration by Partial Fractions

To comprehend the integration procedure using partial fractions, follow the instructions below.

STEP 01 — PROCESS 01

Check whether the given integral is a proper or improper rational function.

STEP 02 — PROCESS 02

If the given function is an improper rational function, identify the type of denominator.

STEP 03 — PROCESS 03

Decompose the integral using a suitable expression by comparing it with the five different forms given above.

STEP 04 — PROCESS 04

Now, divide the integration into parts and integrate the individual functions.

Below is a list of the formulas used to break down improper rational functions.

1. $\int \frac{1}{(ax+b)} dx = \frac{1}{a} ln|ax + b| + C,$
   Where a and c are constant.

2. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

Guideline for Substitution Method

Given the integral, $I = \int f(g(x)) \cdot g'(x)dx$

To substitute, use the following steps :

1. Let $u = g(x)$
2. Differentiate u, $du = g'(x)dx$
3. Substitute $u = g(x)$ and $du = g'(x)dx$
4. Integrate with respect to u.
5. Replace $u = g(x)$ to write the result.

EXAMPLE 1 | INTEGRATION OF LINEAR FACTOR

2.1.1 Here are some of the solved examples to illustrate the method of Integration of Linear Factors by Partial Fractions:

Integrate the following expression using partial fractions: $\int \frac{x+5}{x^2-3x+2}dx$

Solution :

Step 1 to Step 3: Note that the integrand is a proper fraction (because the degree of the numerator is less than the degree of the denominator). As we did in the previous examples on Topic 1 at page 8.

$$= \frac{A}{(x-2)} + \frac{B}{(x-1)} = \frac{7}{(x-2)} - \frac{6}{(x-1)}$$

Step 4 : By decomposing it into two partial fractions, the integral becomes manageable. The integral becomes:

$$= \int \frac{7}{x-2}dx - \int \frac{6}{x-1}dx$$

Formula

$$\int \frac{1}{x}dx = \ln|x| + C, \quad |x| \; must\ be\ positive$$

Answer $= 7\ln(x-2) - 6\ln(x-1) + c$

EXAMPLE 2 | INTEGRATION OF REPEATED LINEAR FACTOR

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax + b)^n$, where n is a positive integer greater than or equal to 2. (Gilbert Strang, 2016)

Integrate the following expression using partial fractions: $\dfrac{x^2+3}{x(x+2)^2}$

Solution :

Step 1 to Step 3: In this example, there is a repeated factor in the denominator. As we did in the previous examples on Topic 1 at page 11.

$$= \frac{A}{x} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

$$= \frac{3}{4x} - \frac{7}{2(x+2)^2} + \frac{1}{4(x+2)}$$

Step 4 : By decomposing it into two partial fractions, the integral becomes manageable. The integral becomes:

$$= \int \frac{3}{4x}dx - \int \frac{7}{2(x+2)^2}dx + \int \frac{1}{4(x+2)}dx$$

$$= \int \frac{7}{2}(x+2)^{-2}dx$$

$$= \frac{7(x+2)^{-2+1}}{2(-2+1)} = \frac{7}{-2(x+2)}$$

Answer:

$$= \frac{3}{4}lnx + \frac{7}{2(x+2)} + \frac{1}{4}ln(x+2) + C$$

EXAMPLE 3 | INTEGRATION OF QUADRATIC FACTOR

You can use the partial fractions technique for functions whose denominators can be factored down to linear factors. However, using this technique is a bit different when there are irreducible quadratic factors.

1. Integrate the following expression using partial fractions: $\dfrac{3x^2-4}{x(x^2+1)}$

Solution :

Step 1 to Step 3: As we did in the previous examples on Topic 1 at page 15..

$$\frac{A(x^2+1)}{x(x^2+1)}+\frac{(Bx+C)x}{x(x^2+1)}=\frac{A}{x}+\frac{(Bx+C)}{(x^2+1)}$$

$$=\frac{4}{x}-\frac{x}{(x^2+1)}$$

Step 4 : By decomposing it into two partial fractions, the integral becomes manageable. The integral becomes:

$$=\int\frac{4}{x}dx-\int\frac{x}{(x^2+1)}dx$$

How to Integrate

For second integral, you use substitution with $u=x^2+1$ and $du=2xdx$; $dx=\frac{du}{2x}$

Answer:

$$=4lnx-\int\frac{x}{u}\times\frac{du}{2x}$$

$$=4lnx-\frac{1}{2}\int\frac{1}{u}du$$

$$=4lnx-\frac{1}{2}\ln u+C$$

$$=4lnx-\frac{1}{2}\ln(x^2+1)+C$$

2. Integrate the following expression using partial fractions: $\int \frac{5x^2+9x-4}{(x+1)(x^2+4)}\,dx$

Solution :

Step 1 to Step 3: As we did in the previous examples on Topic 1 at page 16.

$$= \frac{A}{x+1} + \frac{Bx+C}{(x^2+4)}$$

$$= -\frac{8}{5(x+1)} + \frac{\frac{33}{5}x+\frac{12}{5}}{(x^2+4)}$$

Step 4 : By decomposing it into two partial fractions, the integral becomes manageable. The integral becomes:

$$= \int -\frac{8}{5(x+1)}\,dx + \int \frac{33x}{5(x^2+4)}\,dx + \int \frac{12}{5(x^2+4)}\,dx$$

$$= \int -\frac{8}{5(x+1)}\,dx + \int \frac{33x}{5(x^2+4)}\,dx + \int \frac{12}{5(x^2+2^2)}\,dx$$

How to Integrate?

For second integral, you use substitution with $u = x^2+4$ and $du = 2xdx$; $dx = \frac{du}{2x}$

FORMULA

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Answer: $= -\frac{8}{5}\ln(x+1) + \frac{33}{10}\ln(x^2+4) + \frac{12}{10}\tan^{-1}\frac{x}{2} + C$

DRILL YOUR BRAIN 1

1. Integrate the following functions by the method of partial fractions

$$\int \frac{8x-42}{(x-3)(x+6)}\,dx$$

Solution :

Step 1 to Step 3: As we did in the previous tutorial on Topic 1 at page 21.

Step 4 : Integrate each of the partial fraction form.

Answer: The corrected integrations as well as its partial fraction is

$-2ln(x-3)+ln(x+6)+C$

2. Use partial fraction to find $\int \frac{x^2-2x+3}{(x+1)(x^2+5)} dx$

Solution :

Step 1 to Step 3: As we did in the previous tutorial on Topic 1 at page 22.

Step 4 : Integrate each of the partial fraction form.

Answer: The corrected integrations as well as its partial fraction is

$$ln(x+1) - 2\tan^{-1}\frac{x}{\sqrt{5}} + C$$

3. Use partial fraction to find $\int \frac{5x-3}{(x+2)(x-3)^2}\,dx$

Solution :

Step 1 to Step 3: As we did in the previous tutorial on Topic 1 at page 24

Step 4 : Integrate each of the partial fraction form.

Answer: The corrected integrations as well as its partial fraction is

$$-\frac{13}{25}\ln(x+2)-\frac{12}{5(x-3)}+\frac{13}{25}\ln(x-3)+C$$

EXAMPLE 4 | INTEGRATION OF IMPROPER FRACTION

2.1.2 This problem is an example of an improper rational function. This rational function is improper because its numerator has a degree that is higher than its denominator.

Integrate the following expression using partial fractions:

$$\int \frac{3x^3 - 2x^2 + 4x + 7}{x^2 + 2x} dx$$

Solution :

Step 1 to Step 3: The degree of the numerator is greater than the degree of the numerator. This is improper fraction. As we did in the previous examples on Topic 1 at page 29

$$= \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)} = \frac{A}{x} + \frac{(Bx+C)}{(x^2+1)}$$

$$= \frac{7}{2x} + \frac{33}{2(x+2)}$$

Step 4 : By decomposing it into two partial fractions, the integral becomes manageable. The integral becomes:

$$\int \frac{7}{2x} + \frac{33}{2(x+2)} dx$$

Answer: $= \frac{7}{2} lnx + \frac{33}{2} \ln(x+2) + C$

DRILL YOUR BRAIN 2

Integrate the following functions by the method of partial fractions.

$$\int \frac{x^2}{x^2 + x - 2} dx$$

Solution :

Step 1 to Step 3: As we did in the previous tutorial on Topic 1 at page 31

Step 4 : Integrate each of the partial fraction form.

Answer: The corrected integrations as well as its partial fraction is $x - \frac{4}{3} ln(x + 2) + \frac{1}{3} ln(x - 1) + C$

PRACTICE YOUR KNOWLEDGE 2

LET'S DO IT

Write the given expression in the form of partial fraction and then integrate with respect to x :

1. $\int \frac{2x+1}{x^3+x^2-2x}dx$
2. $\int \frac{2z+3}{z^2(4z+1)}dz$
3. $\int \frac{5x^2+30x+43}{(x+3)^3}dx$
4. $\int \frac{2x^3+10x}{(x^2+1)^2}dx$
5. $\int \frac{6x^2-3x}{(x-2)(x+4)}dx$
6. $\int \frac{-2x+4}{(x^2+1)(x-1)}dx$
7. $\int \frac{4x^3+10x+4}{x(2x+1)}dx$
8. $\int \frac{x-2}{(2x-1)^2(x-1)}dx$
9. $\int \frac{3x}{(x+3)(2x+1)}dx$
10. $\int \frac{x^3+2}{x^2+2}dx$
11. $\int \frac{4x^2+7x-3}{(x^2-7x+12)(x+5)}dx$
12. $\int \frac{3x+2}{x^2-9}dx$
13. $\int \frac{9}{(x+1)(x-2)^2}dx$
14. $\int \frac{2+5x+x^2}{x(x^2+1)}dx$
15. $\int \frac{4x-10}{x^2-2x-3}dx$
16. $\int \frac{x^2+8}{x^2-2x+6}dx$
17. $\int \frac{x}{5x^2-14}dx$
18. $\int \frac{2x^2+x+2}{(x-1)(x^2+9)}dx$
19. $\int \frac{x^2-2x+1}{x(2x-1)(x+2)}dx$
20. $\int \frac{15}{x^3-9x}dx$

LET'S CHECK YOUR ANSWER!

SCAN ME!

PARTIAL FRACTION IN INVERSE LAPLACE TRANSFORMS

3

CHAPTER OUTCOMES

At the end of this chapter, you should be able to

- **Solve the Inverse Laplace Transforms by Using Partial Fraction Method**

ChapTer 3 : ParTial FracTion In InVerse LapLace TransForms

3.0 INTRODUCTION

In mathematics, the word 'transform' has the same meaning as in everyday language. The Laplace Transform is one of number of integral transforms used by engineers. The Laplace Transforms can be used to solve a linear constant-coefficient differential equation by transforming it into an algebraic equation. The resulting algebraic equation is solved and then the transform is reserved to find the solution of the differential equation in terms of the original variable.

The Laplace Transforms can also be used to calculate transfer functions. These functions describe the elements of an engineering system and are particularly important in the design of control system.

3.1 METHOD TO SOLVE LAPLACE TRANSFORM

There are 3 common methods to solve the Laplace Transform:

1) by using Definition of The Laplace Transform
2) by using Table of Laplace Transform
3) by using Properties of The Laplace Transform
    a. Linearity Property

b. First Shift Theorem

c. Multifaction With t^n

This chapter is not going to explain about the details of Inverse Laplace Transform. Our main concern is how to solve the problem of Inverse Laplace Transform by using the Partial Fraction Method.

3.2 METHOD TO SOLVE THE INVERSE LAPLACE TRANSFORM

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function;

$$F\ (s) = \frac{P\ (s)}{Q\ (s)}$$

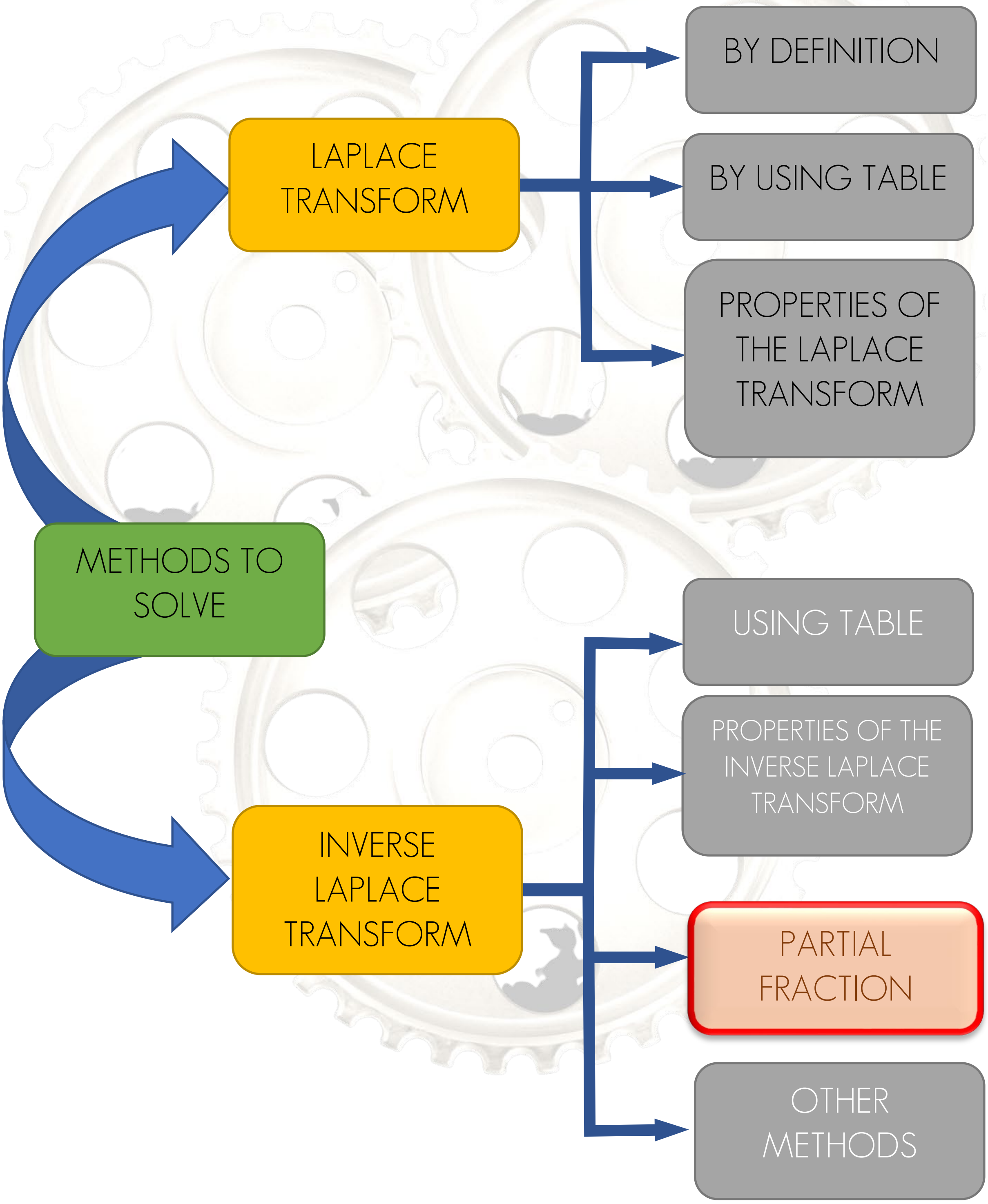
If $F\ (s)$ is a Laplace transform, we need only consider the case where degree (P) < degree (Q) (Murray, S. R. , 1965). Finding the Laplace transform of a function is not terribly difficult if we have got a table of transforms. In this chapter, we want to find the Inverse Laplace Transform of $F\ (s)$ and use the following notation;

$$f\ (t) = L^{-1}\{F(s)\}$$

The Inverse Laplace Transform problem can be solved by:

1) Using Table to Find Inverse Laplace Transform
2) The Properties of the Inverse Laplace Transform
    a) Linear Property
    b) First Shift Theorem
    c) Second Shift Theorem

3) Partial Fraction Method
4) Method of Differential Equations
5) Others Method : Miscellaneous Method, Heaviside Expansion Formula, Beta Function and Evaluation Of Integrals



From the following formula, we can identify the various important properties of Inverse Laplace Transform:

Table of Inverse Laplace Transforms

	$F(t)$	$L\{F(t)\} = f(s)$	$L^{-1}\{f(s)\} = F(t)$
1	1	$\frac{1}{s}$ $\quad s > 0$	1
2	t	$\frac{1}{s^2}$ $\quad s > 0$	t
3	t^n $n = 0, 1, 2, \dots$	$\frac{1}{s^{n+1}}$ $\quad s > 0$ $n = 0, 1, 2, \dots.$ Note: Factorial $n = n! = 1 \cdot 2 \cdots n$ By definition, $0! = 1$	$\frac{t^n}{n!}$
4	e^{at}	$\frac{1}{s-a}$ $\quad s > a$	e^{at}
5	$\sin bt$	$\frac{1}{s^2 + b^2}$ $\quad s > 0$	$\frac{\sin bt}{b}$
6	$\cos bt$	$\frac{s}{s^2 + b^2}$ $\quad s > 0$	$\cos bt$
7	$\sinh bt$	$\frac{1}{s^2 - b^2}$ $\quad s > \lvert b \rvert$	$\frac{\sinh bt}{b}$
8	$\cosh bt$	$\frac{s}{s^2 - b^2}$ $\quad s > \lvert b \rvert$	$\cosh bt$

EXAMPLE 1 | LINEAR FACTOR

Write the following expression in the form of partial fraction.

$$\frac{1}{s\,(s-1)}$$

Next, obtain the Inverse Laplace Transform of

$$L^{-1}\left\{\frac{1}{s\,(s-1)}\right\}$$

Solution:

Step 1 : Transform to linear factor form : $\frac{A}{(ax+b)} + \frac{B}{(cx+d)}$

$$\frac{1}{s\,(s-1)} = \frac{A}{s} + \frac{B}{(s-1)}$$

Step 2 : Multiply each of numerator to make the denominator equal..

$$= \frac{A}{s} \times \frac{(s-1)}{(s-1)} + \frac{B}{(s-1)} \times \frac{s}{s}$$

$$= \frac{A(s-1)+Bs}{s(s-1)}$$

Step 3 : Find the value of A and B

Method 1 | Elimination

$$1 = A(s-1) + Bs$$

This must be true for any value of s

If $s-1=0, s=1$

$1 = A(1-1) + B(1)$

$\boldsymbol{B = 1}$

If $s = 0,$

$1 = A(0-1) + B(0)$

$\boldsymbol{A = -1}$

Method 2 | Expand, equate the coefficient and solve by using calculator.

Expand

$$1 = A(s-1) + B(s)$$

$$1 = As - A + Bs$$

By equating the coefficients of x and C (constant)

s	C
$A + B = 0$	$-A = 1$

$$1A + 1B = 0 \qquad -1A = 1$$

$$a_1 \quad b_1 \quad c_1 \qquad a_2 \quad c_2$$

The values A and B by using a Calculator.

$$\boldsymbol{A = x = -1}$$

$$\boldsymbol{B = y = 1}$$

$$\frac{1}{s\,(s-1)} = \frac{-1}{(x-2)} + \frac{1}{(x-1)}$$

Step 4 : Solve the Inverse Laplace Transforms

The result is easier to be solved if it is extended to two functions:

$$L^{-1}\left\{\frac{1}{s\,(s-1)}\right\} = L^{-1}\left\{\frac{-1}{s}\right\} + L^{-1}\left\{\frac{1}{(s-1)}\right\}$$

Using Table:

$a = constant$

If $L\{f(t)\} = F(s)$

$L\{a\} = \frac{a}{s}$

$\therefore L^{-1}\left\{\frac{a}{s}\right\} = a$

Using Table:

If $L\{f(t)\} = e^{\pm at}$

$L\{e^{\pm at}\} = \frac{1}{s \mp a}$

$\therefore L^{-1}\left\{\frac{1}{s \mp a}\right\} = e^{at}$

Linear Property

$\left\{\frac{-1}{s}\right\}$

$a = -1$

$\left\{\frac{1}{(s-1)}\right\}$

$a = 1$

$$= L^{-1}\left\{\frac{-1}{s}\right\} + e^{at}L^{-1}\left\{\frac{1}{(s-1)}\right\}$$

$$= (-1) + e^{(1)t}(1)$$

Answer

Thus, $\boldsymbol{L^{-1}\left\{\frac{1}{s\,(s-1)}\right\} = -1 - e^{t}}$

EXAMPLE 2 | LINEAR FACTOR

Let's do a couple more examples to remind you how to solve Inverse Laplace Transforms problems.

Use partial fraction to find $L^{-1}\left\{\frac{2s+3}{(s+3)(s+1)}\right\}$

Solution :

Step 1: Transform to linear factor form : $\frac{A}{(x+a)} + \frac{B}{(x+b)}$

$$\frac{2s+3}{(s+3)(s+1)} = \frac{A}{(s+3)} + \frac{B}{(s+1)}$$

Step 2 : Multiply each of numerator to make the denominator equal..

$$= \frac{A}{(s+3)} \times \frac{(s+1)}{(s+1)} + \frac{B}{(s+1)} \times \frac{(s+3)}{(s+3)}$$

Step 3 : Find the value of A and B

Method 1 | Elimination

$$2s+3 = A(s+1) + B(s+3)$$

This must be true for any value of s

If $s+1=0, s=-1$

$2(-1)+3 = A(-1+1) + B(-1+3)$

$$\boldsymbol{B = \frac{1}{2}}$$

If $s+3=0, s=-3$

$2(-3)+3 = A(-3+1) + B(-3+3)$

$$\boldsymbol{A = \frac{3}{2}}$$

Answer Thus, $\boldsymbol{\frac{2s+3}{(s+3)(s+1)} = \frac{3}{2(s+3)} + \frac{1}{2\,(s+1)}}$

Step 4 : Solve the Inverse Laplace Transforms

The result is easier to be solved if it is extended to two functions:

$$L^{-1}\left\{\frac{2s+3}{(s+3)(s+1)}\right\} = L^{-1}\left\{\frac{3}{2(s+3)}\right\} + L^{-1}\left\{\frac{1}{2\ (s+1)}\right\}$$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{(s+3)}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)}\right\}$$

Linear Property

$\left\{\frac{1}{(s+3)}\right\}$ $a = -3$

$\left\{\frac{1}{(s+1)}\right\}$ $a = -1$

Using Table:

If $L\{f(t)\} = e^{\pm at}$

$$L\{e^{\pm at}\} = \frac{1}{s \mp a}$$

$$\therefore L^{-1}\left\{\frac{1}{s \mp a}\right\} = e^{at}$$

$$= \frac{3}{2}e^{(-3)t} + \frac{1}{2}e^{(-1)t}$$

Answer Thus,

$$L^{-1}\left\{\frac{2s+3}{(s+3)(s+1)}\right\} = \frac{3}{2}e^{-3t} + \frac{1}{2}e^{-t}$$

DRILL YOUR BRAIN 1

By using partial fraction method, find the Inverse Laplace Transform of the following.

a) $\dfrac{2s+1}{(s-2)(s+3)(s-1)}$

Solution :

Step 1: Transform to linear factor form $\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$

Step 2 : Multiply each of numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 1 | Elimination

Answer Here's the partial fraction decomposition for this part

$$\frac{2s+1}{(s-2)(s+3)(s-1)} = \frac{1}{(s-2)} - \frac{\frac{1}{4}}{(s+3)} - \frac{\frac{3}{4}}{(s-1)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{2s+1}{(s-2)(s+3)(s-1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-2)}\right\} - L^{-1}\left\{\frac{\frac{1}{4}}{(s+3)}\right\} - L^{-1}\left\{\frac{\frac{3}{4}}{(s-1)}\right\}$$

Answer The corrected transforms as well as its inverse transform is

$$L^{-1}\left\{\frac{2s+1}{(s-2)(s+3)(s-1)}\right\} = e^{2t} - \frac{1}{4}e^{-3t} - \frac{3}{4}e^{t}$$

b) $\frac{s}{(s-1)(s^2-4)}$

Solution :

Step 1: Transform to linear factor form $\frac{A}{(x+a)}+\frac{B}{(x+b)}+\frac{C}{(x+c)}$

Step 2 : Multiply each of the numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 1 | Elimination

Answer Here's the partial fraction decomposition for this part

$$\frac{s}{(s-1)(s^2-4)} = -\frac{\frac{1}{3}}{(s-1)} - \frac{\frac{1}{2}}{(s-2)} - \frac{\frac{1}{6}}{(s+2)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{s}{(s-1)(s^2-4)}\right\}$$

$$= -L^{-1}\left\{\frac{\frac{1}{3}}{(s-1)}\right\} - L^{-1}\left\{\frac{\frac{1}{2}}{(s-2)}\right\} - L^{-1}\left\{\frac{\frac{1}{6}}{(s+2)}\right\}$$

Answer

The corrected transforms as well as its inverse transform is

$$\boldsymbol{L}^{-1}\left\{\frac{s}{(s-1)(s^2-4)}\right\} = \frac{\mathbf{1}}{\mathbf{4}}e^{t} + \frac{\mathbf{1}}{\mathbf{2}}e^{2t} - \frac{\mathbf{1}}{\mathbf{6}}e^{-2t}$$

EXAMPLE 3 | REPEATED LINEAR FACTOR

Find Inverse Laplace Transforms for $F(s) = \frac{2s^2+1}{(s+1)(s+2)^2}$ by using partial fraction method.

Solution :

Step 1: Transform to linear factor form $\frac{A}{(ax+b)} + \frac{B}{(cx+d)^2} + \frac{C}{(cx+d)}$

$$\frac{2s^2+1}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

Step 2 : Multiply each of the numerator to make the denominator equal..

$$= \frac{A}{(s+1)} \times \frac{(s+3)^2}{(s+3)^2} + \frac{B}{(s+2)^2} \times \frac{(s+1)}{(s+1)} + \frac{C}{(s+2)} \frac{(s+1)(s+2)}{(s+1)(s+2)}$$

Step 3 : Find the value of A, B and C

Method 1 | Elimination

$$2s^2 + 1 = A(s+2)^2 + B(s+1) + C(s+1)(s+2)$$

If $s + 1 = 0, s = -1$

$2(-1)^2 + 1 = A(-1+2)^2 + B(-1+1) + C(-1+1)(-1+2)$

$\boldsymbol{A = 3}$

If $s + 2 = 0, s = -2$

$2(-2)^2 + 1 = A(-2+2)^2 + B(-2+1) + C(-2+1)(-2+2)$

$\boldsymbol{B = -9}$

This must be true for any value of s

$$2s^2 + 1 = A(s+2)^2 + B(s+1) + C(s+1)(s+2)$$

Choose any value of s except s = -1 and s = -2
Substitute the value of A and B

If $s = 0$,

$$2(0)^2 + 1 = (3)(0+2)^2 + (-9)(0+1) + C(0+1)(0+2)$$

$$\boldsymbol{C = -1}$$

Answer

$$\frac{2s^2+1}{(s+1)(s+2)^2} = \frac{3}{(s+1)} + \frac{(-9)}{(s+2)^2} + \frac{(-1)}{(s+2)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{2s^2+1}{(s+1)(s+2)^2}\right\}$$

$$= L^{-1}\left\{\frac{\mathbf{3}}{(s+1)}\right\} + L^{-1}\left\{\frac{\mathbf{(-9)}}{(s+2)^2}\right\} + L^{-1}\left\{\frac{\mathbf{(-1)}}{(s+2)}\right\}$$

Using Table:

If $L\{f(t)\} = e^{\pm at}$

$$L\{e^{\pm at}\} = \frac{1}{s \mp a}$$

$$\therefore L^{-1}\left\{\frac{1}{s \mp a}\right\} = e^{at}$$

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{at\} = \frac{a}{s^2}$$

$$\therefore L^{-1}\left\{\frac{a}{s^2}\right\} = at$$

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{a\} = \frac{a}{s}$$

$$\therefore L^{-1}\left\{\frac{a}{s}\right\} = a$$

$$= 3\ L^{-1}\left\{\frac{1}{(s+1)}\right\} - 9L^{-1}\left\{\frac{1}{(s+2)^2}\right\} - L^{-1}\left\{\frac{(1)}{(s+2)}\right\}$$

1st Shift Theorem

$\left\{\frac{1}{(s+1)}\right\}$ $a = -1$

$\left\{\frac{1}{(s+2)}\right\}$ $a = -2$

$$= 3\ e^{-t}L^{-1}\left\{\frac{1}{s}\right\} - 9e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} - e^{-2t}L^{-1}\left\{\frac{1}{s}\right\}$$

Answer Thus, $L^{-1}\left\{\frac{2s^2+1}{(s+1)(s+2)^2}\right\} = 3\ e^{-t} - 9e^{-2t}t - e^{-2t}$

DRILL YOUR BRAIN 2

Find Inverse Laplace Transform for $F(s) = \frac{16s^2+28s+19}{(s+2)(2s+1)^2}$ by using partial fraction method.

Solution :

Step 1: Transform to linear factor form $\frac{A}{(ax+b)} + \frac{B}{(cx+d)^2} + \frac{C}{(cx+d)}$

$$\frac{16s^2+28s+19}{(s+2)(2s+1)^2} = \frac{A}{(s+2)} + \frac{B}{(2s+1)^2} + \frac{C}{(2s+1)}$$

Step 2 : Multiply each of numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 1 | Elimination

Method 2 || Expand & Compare the Coefficients

Answer Here's the partial fraction decomposition for this part

$$\frac{16s^2+28s+19}{(s+2)(2s+1)^2} = \frac{3}{(s+2)} + \frac{6}{(2s+1)^2} + \frac{2}{(2s+1)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Answer The corrected transforms as well as its inverse transform is

$$L^{-1}\left\{\frac{16s^2+28s+19}{(s+2)(2s+1)^2}\right\} = \mathbf{3\,e^{-2t} + \frac{3}{2}te^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}}$$

EXAMPLE 4 | QUADRATIC FACTOR

By using partial fraction method, find the Inverse Laplace Transform for

$$\frac{1}{(s+1)(s^2+1)}$$

Solution :

Step 1: Transform to linear factor form $\frac{A}{(x+a)}+\frac{Bx+C}{(bx^2+cx+d)}$

$$\frac{1}{(s+1)(s^2+1)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+1)}$$

Step 2 : Multiply each of numerator to make the denominator equal..

$$=\frac{A}{(s+1)}\times\frac{(s^2+1)}{(s^2+1)}+\frac{Bs+C}{(s^2+1)}\times\frac{(s+1)}{(s+1)}$$

If the quadratic equation can't be factorized, continue the step by choosing the Quadratic Factor

Step 3 : Find the value of A, B and C

Method 1 | Elimination

$$1 = A(s^2+1) + (Bs+C)(s+1)$$

If $s+1=0, s=-1$

$$1 = A(s^2+1) + (B(-1)+C)(-1+1)$$

$$A=\frac{1}{2}$$

This must be true for any value of s

If $s=0$,

$$1=\frac{1}{2}(0+1)^2+\left(\left(\frac{1}{2}\right)(0)+\frac{1}{2}\right)(0+1)$$

$$C=\frac{1}{2}$$

If $s = 1$,

$$1 = \frac{1}{2}(s^2 + 1)^2 + \left(Bs + \frac{1}{2}\right)(s + 1)$$

$$B = -\frac{1}{2}$$

Choose any value of s except s = -1 and s = 1
Substitute the value of A and C

Answer

$$\frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{(s+1)} + \frac{\left(-\frac{1}{2}\right)s+\frac{1}{2}}{(s^2+1)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Using Table:

If $L\{f(t)\} = e^{\pm at}$

$$L\{e^{\pm at}\} = \frac{1}{s \mp a}$$

$$\therefore L^{-1}\left\{\frac{1}{s \mp a}\right\} = e^{at}$$

Extended to three functions:

$$L^{-1}\left\{\frac{1}{(s + 1)(s^2 + 1)}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{2}}{(s+1)}\right\} + L^{-1}\left\{\frac{\left(-\frac{1}{2}\right)s+\frac{1}{2}}{(s^2+1)}\right\}$$

$$\left\{\frac{1}{(s + 1)}\right\}$$

$$a = -1$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{s}{(s^2+1)}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s^2+1)}\right\}$$

1st Shift Theorem

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{\sin t\} = \frac{s}{(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+1)}\right\} = \cos\ bt$$

$$\left\{\frac{s}{(s^2+1)}\right\}$$

$$b = 1$$

$$\left\{\frac{1}{(s^2+1)}\right\}$$

$$b = 1$$

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{\sin t\} = \frac{1}{(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin\ bt$$

Answer Thus, $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$

Let's do some harder problems.

EXAMPLE 5 | QUADRATIC FACTOR With Completing The Square

Find the Inverse Laplace Transform of $\frac{s-1}{(s-2)(s^2-2s+10)}$

Solution :

Step 1: Transform to linear factor form $\frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$

$$\frac{s-1}{(s-2)(s^2-2s+10)} = \frac{A}{(s-2)} + \frac{Bs+C}{s^2-2s+10}$$

If the quadratic equation cannot be factorized, continue the step by choosing the Quadratic Factor

Step 2 : Multiply each of the numerator to make the denominator equal..

$$= \frac{A}{(s-2)} \times \frac{s^2-2s+10}{s^2-2s+10} + \frac{Bs+C}{s^2-2s+10} \times \frac{(s-2)}{(s-2)}$$

Step 3 : Find the value of A, B and C

Method 1 | Elimination

$$s - 1 = A(s^2 - 2s + 10) + (Bs + C)(s - 2)$$

If $s - 2 = 0, s = 2;$

$2 - 1 = A(2^2 - 2(2) + 10) + (B(2) + C)(2 - 2)$

$\boldsymbol{A = \frac{1}{10}}$

This must be true for any value of s

If $s = 0;$

$0 - 1 = \frac{1}{10}(0^2 - 2(0) + 10) + (B(0) + C)(0 - 2)$

$\boldsymbol{C = 1}$

Choose any value of s except s = -2 and s = 0
Substitute the value of A and C

If $s = 1;$

$1 - 1 = \frac{1}{10}(1^2 - 2(1) + 10) + (B(1) + 1)(1 - 2)$

$\boldsymbol{B = -\frac{1}{10}}$

Answer

$$\frac{s-1}{(s-2)(s^2-2s+10)} = \frac{\frac{1}{10}}{(s-2)} + \frac{\left(-\frac{1}{10}\right)s + 1}{s^2 - 2s + 10}$$

Step 4 : Solve the Inverse Laplace Transforms

$$L^{-1}\left\{\frac{s-1}{(s-2)(s^2-2s+10)}\right\} = L^{-1}\left\{\frac{\frac{1}{10}}{(s-2)} + \frac{\left(-\frac{1}{10}\right)s+1}{s^2-2s+10}\right\}$$

Completing the square: $\frac{\left(\frac{1}{10}\right)s+1}{s^2-2s+10}$

$$s^2 - 2s + 10 = 0; a = 1, b = -2, c = 10$$

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

$$= \left(s + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

$$= \left(s + \frac{(-2)}{2}\right)^2 + 10 - \left(\frac{-2}{2}\right)^2$$

$$= (s-1)^2 + 9$$

$$= (s-1)^2 + 3^2$$

$$\therefore \frac{\left(\frac{1}{10}\right)s+1}{s^2-2s+10} = \frac{\left(\frac{1}{10}\right)s+1}{(s-1)^2+3^2}$$

The result is easier to be solved if it is extended to three functions:

$$\therefore L^{-1}\left\{\frac{s-1}{(s-2)(s^2-2s+10)}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{10}}{(s-2)} + \frac{\left(-\frac{1}{10}\right)s+1}{s^2-2s+10}\right\}$$

$$= \frac{1}{10}L^{-1}\left\{\frac{1}{(s-2)}\right\} - \frac{1}{10}L^{-1}\left\{\frac{s}{(s-1)^2+3^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2+3^2}\right\}$$

1st Shift Theorem

Using Table:

If $L\{f(t)\} = e^{\pm at}$

$$L\{e^{\pm at}\} = \frac{1}{s \mp a}$$

$$\therefore L^{-1}\left\{\frac{1}{s \mp a}\right\} = e^{at}$$

$$\left\{\frac{1}{(s-2)}\right\}$$

$a = 2$

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{\sin t\} = \frac{s}{(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+1)}\right\} = \cos\ bt$$

$$-\frac{1}{10}L^{-1}\left\{\frac{(s-1)}{(s-1)^2+3^2}\right\}$$

$$\left\{\frac{(s-1)}{(s-1)^2+3^2}\right\} = \left\{\frac{s}{s^2+b^2}\right\}$$

$a = 1$

$b = 3$

Using Table:

If $L\{f(t)\} = F(s)$

$$L\{\sin t\} = \frac{1}{(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin\ bt$$

$$L^{-1}\left\{\frac{1}{(s-1)^2+3^2}\right\}$$

$$\left\{\frac{1}{3}\left(\frac{3}{(s-1)^2+3^2}\right)\right\} = \left\{\frac{1}{s^2+b^2}\right\}$$

$a = 1$

$b = 3$

Answer Thus,

$$L^{-1}\left\{\frac{s-1}{(s-2)(s^2-2s+10)}\right\} = \frac{1}{10}e^{2t} - \frac{1}{10}e^{t}\cos 3t + \frac{1}{3}e^{t}\sin 3t$$

DRILL YOUR BRAIN 3

By using partial fraction method, find the Inverse Laplace Transform for

a) $\dfrac{4s^2-5s+11}{(s+1)(s^2+9)}$

Solution :

Step 1: Transform to linear factor form $\dfrac{A}{(ax+b)}+\dfrac{Bx+C}{(cx^2+d)}$

Step 2 : Multiply each of the numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 2 | Expand and Compare the Coefficients:

Answer Here's the partial fraction decomposition for this part

$$\frac{4s^2-5s+11}{(s+1)(s^2+9)} = \frac{2}{(s+1)} + \frac{2s-7}{(s^2+9)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{4s^2-5s+11}{(s+1)(s^2+9)}\right\}$$

Answer The corrected transforms as well as its inverse transform is

$$L^{-1}\left\{\frac{4s^2-5s+11}{(s+1)(s^2+9)}\right\} = 2\,e^{-t} + 2\cos 3t - \frac{7}{3}\sin 3t$$

b) $\frac{15s}{(s+3)(s^2+1)}$

Solution :

Step 1: Transform to linear factor form $\frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$

Step 2 : Multiply each of the numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 1 | Elimination

Method 2 | Expand and Compare the Coefficients:

Answer Here's the partial fraction decomposition for this part

$$\frac{15s}{(s+3)(s^2+1)} = -\frac{\frac{9}{2}}{(s+3)} + \frac{\frac{9}{2}s+\frac{3}{2}}{(s^2+1)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{15s}{(s+3)(s^2+1)}\right\}$$

Answer The corrected transforms as well as its inverse transform is

$$\boldsymbol{L}^{-1}\left\{\frac{15s}{(s+3)(s^2+1)}\right\} = -\frac{9}{2}\,e^{-3t} + \frac{9}{2}\cos t - \frac{3}{2}\sin t$$

c) $\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}$

Solution :

Step 1: Transform to linear factor form $\dfrac{A}{(ax+b)}+\dfrac{Bx+C}{(cx^2+d)}$

Step 2 : Multiply each of the numerator to make the denominator equal..

Step 3 : Find the value of A, B and C

Method 1 | Elimination

Method 2 | Expand and Compare the Coefficients:

Answer Here's the partial fraction decomposition for this part

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)}=\frac{3}{(s+1)}+\frac{s-6}{(s^2+4)}$$

Step 4 :

Solve the Inverse Laplace Transforms

Extended to three functions:

$$L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$$

Answer The corrected transforms as well as its inverse transform is

$$\boldsymbol{L}^{-1}\left\{\frac{4s^2-5s+6}{(s+1)\left(s^2+4\right)}\right\}=\mathbf{3}\,e^{-t}+\cos 2t+\mathbf{3}\sin 2t$$

Partial fractions are a fact of life to solve Inverse Laplace Transforms problems (Paul D., 2022). Make sure that we can deal with it.

PRACTICE YOUR KNOWLEDGE 3

LET'S DO IT

a) Find the Inverse Laplace Transform of the followings by using partial fraction method :

1. $F(s) = \dfrac{s}{(s-1)(s+2)}$
2. $F(s) = \dfrac{5s-4}{s^2-s-2}$
3. $F(s) = \dfrac{s^2+1}{s(s+1)(s-1)}$
4. $F(s) = \dfrac{s-2}{s^2(s-1)^2}$
5. $F(s) = \dfrac{3s^2-s-1}{(s-2)(s+1)^2}$
6. $F(s) = \dfrac{5}{s^2-100}$
7. $F(s) = \dfrac{5s^2+8s-1}{(s+3)(s^2+1)}$
8. $F(s) = \dfrac{s}{(s-3)(s+6)^2}$
9. $F(s) = \dfrac{2}{s(s^2+4)}$
10. $F(s) = \dfrac{s}{(s+1)(s^2+1)}$
11. $F(s) = \dfrac{3s^2-4}{s(s^2+1)}$
12. $F(s) = \dfrac{2s-1}{s^2-5s+2}$

b) Find:

1. $L^{-1}\left\{\dfrac{3s+7}{s^2-s-2}\right\}$
2. $L^{-1}\left\{\dfrac{3s^2-4}{(s+1)(s-2)(s-3)}\right\}$
3. $L^{-1}\left\{\dfrac{5s^2-15s-11}{(s+1)(s-2)(s-3)}\right\}$
4. $L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$
5. $L^{-1}\left\{\dfrac{1}{s^2+3s-10}\right\}$
6. $L^{-1}\left\{\dfrac{4s-1}{(s-1)(s^2-1)}\right\}$
7. $L^{-1}\left\{\dfrac{1}{s^2-7s+12}\right\}$
8. $L^{-1}\left\{\dfrac{s}{(s-1)(s+2)^2}\right\}$

LET'S CHECK YOUR ANSWER!

SCAN ME!

RefereNces

References

Abramson, J. (2021). *College Algebra 2e.* OpenStax.

Dawkins, P. (2022). *Differential Equations.* Retrieved from https://tutorial.math.lamar.edu/classes/de/de.aspx

Gilbert Strang, E. ". (2016). In *Calculus Volume 2.* Texas: OpenStax.

Krantz, S. G. (2003). *Calculus DeMYSTiFied, A Self-Teaching Guide.* United States of America: McGraw Hill.

Mun, C., Sook King, T., & Ying Juan, A. (2023). *Mathematics for Matriculation Science.* SAP Publications (M) Sdn. Bhd.

Siak Peng, C., Pei Eng, C., & Pei Cheng, C. (2005). *Mathematics for Matricuation 1.* Selangor: Oriental Academic Publication.

Spiegel, M. R. (1965). *Laplace Transform.* United States of America: McGraw Hill.

Stewart, J. (2006). *Single Variable Calculus Concept & Contexts 3.* Bob Pirtle.

Yang, C. (2020). *Diagrams Mathematics for Matriculation (Science).* SAP Publications (M) Sdn. Bhd.

Yang, C. (2021). *Mathematics for Matriculation Semester 1 (Science).* SAP Publications (M) Sdn. Bhd.

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