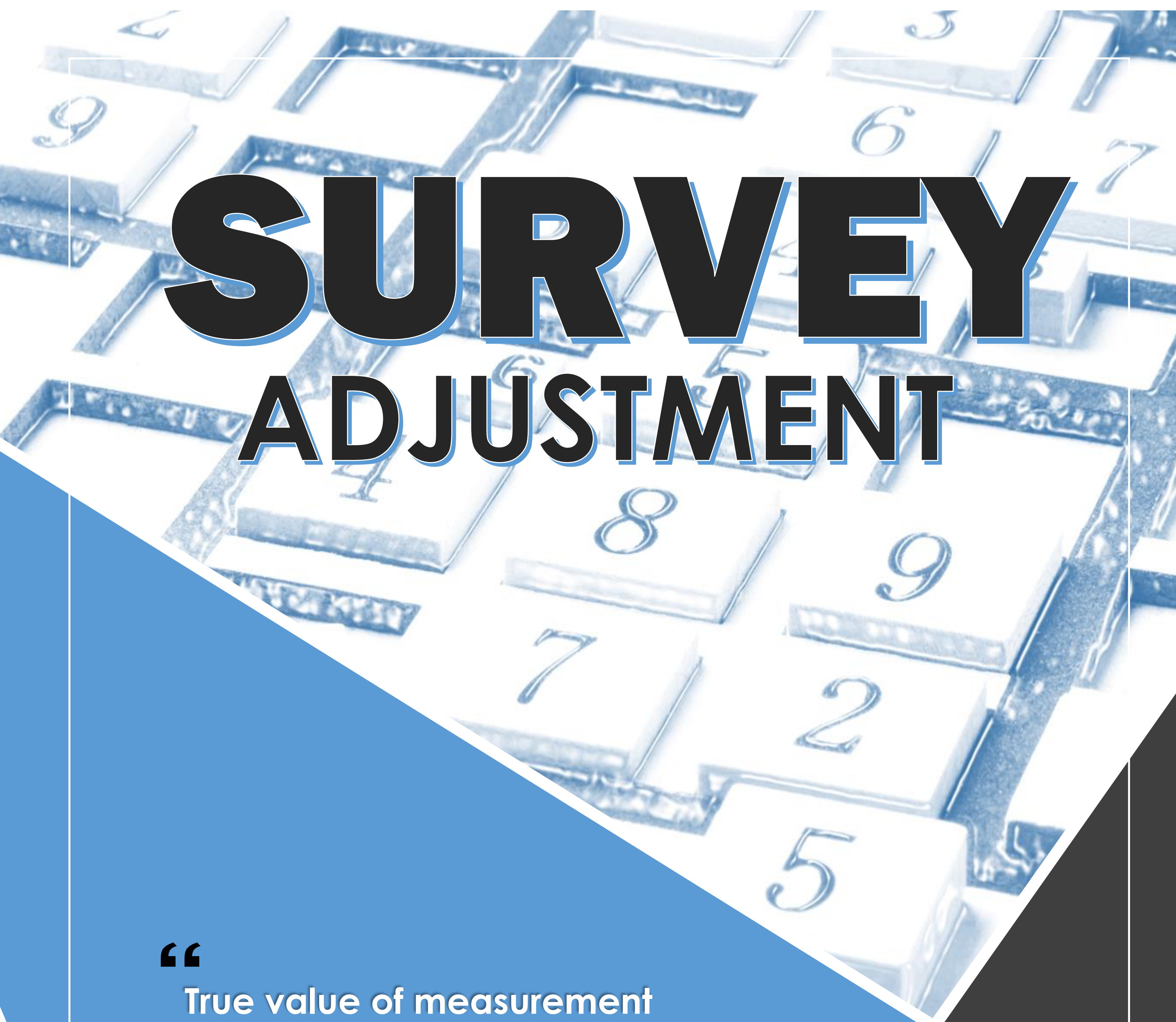

SURVEY
ADJUSTMENT

"True value of measurement is unknown,
Every observation contains error"

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SURVEY

ADJUSTMENT

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Published in 2021

Perpustakaan Negara Malaysia Cataloguing-in-Publication Data

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SURVEY ADJUSTMENT / NOOR FAIZAH BINTI ZOHARDIN
Mode of access: Internet
eISBN 978-967-2241-83-6
1.Surveying
2.Goverment publications—Malaysia.
3. Electronic book.
I. Title.
526.9

Published by:
Politeknik Merlimau, Melaka
KB1031 Pej Pos Merlimau,
77300 Merlimau Melaka

PREFACE

SURVEY ADJUSTMENT provides the students with knowledge on adjustment. The book emphasizes the calculation of adjustment using the least square adjustment method through observation and condition equations in solving surveyed data. Besides, it is also provides students with knowledge and practical skills to calculate and adjust surveyed data.

This book is written specifically to satisfy the syllabus requirements for subject DCG 50192 Survey Adjustment. This book contains all required topics for Diploma Geomatic.

This book contains 5 chapter that have been planned and arranged carefully base on the syllabus of Polytechnic Malaysia. All concepts for each topic are accompanied by detail explanations, followed by example and complete solutions.

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Chapter 1 : INTRODUCTION TO SURVEY ADJUSTMENT

This topic describes the purposes of survey adjustment distinguish the mathematical and functional models from the statistical model

Chapter 2 : STATISTICAL SAMPLE

This topic explains the measurement of central tendency and measurement of dispersion and how the matrix variance covariance is derived,

Chapter 3 : VARIANCE- COVARIANCE PROPAGATION

This topic focuses on the calculation of variance-covariance propagation, derivative formula variance-covariance propagation for linear functions, non -linear functions. Solve the partial differential calculation. Application of the variance covariance propagation calculation linear case and nonlinear cases.

Chapter 4 : WEIGHT OF OBSERVATION

This topic focuses on the calculation of weight of observation, the concept of weight in survey observation.

Chapter 5 : LEAST SQUARE ADJUSTMENT APPLICATIONS

This topic demonstrates the steps in solving the Least Square, method of equation, the concept of Least Square Adjustment, how the Normal Equation is derived, the principles of Least Square and how the variance-covariance matrix for the parameter X is calculated.

TABLE OF CONTENTS

“Good measurement required a combination of human skill and mechanical equipment”

Chapter 1: Introduction to Survey Adjustment

ADJUSTMENT

- Adjustment is a process of making measured values of a quantity more accurate before they are used in the computations for the determination of points position that are associated with the measurements
- The method of estimating and distributing random errors in the observed values in order to make it conform to certain geometrical conditions, hence the resulted/adjusted values are known as the most probable values for the quantity involves.

PURPOSE OF SURVEY ADJUSTMENT

- To make sure final survey value accurate and close to the truth as possible
- To evaluate and measure the confidence in result
- To determine how accurate each value is
- To estimating and distributing random errors in the observed values
- To reduce error size when making measurement
- To analyst the error and adjust the data
- To analysing and adjusting survey data
- To identify the Accuracy standard for survey obtained from least square adjustment

MATHEMATICAL MODEL

Table 1.1 : Mathematical Model

FUNCTIONAL MODEL	STOCHASTIC MODEL
Adjustment computations is an equation or set of equations/functions that represents or defines an adjustment condition	The determination of variances, and subsequently the weights of the observations
To describe a system or physical condition	To describe probability variable like observation
Equations used in modeling observations [observation & condition equations]; express geometrical relationship between observations (distance) & parameters (coordinates)	Describes random/stochastic property of observations in the form of weight [standard deviation] of observation, controls weights of observations [control the correction to an observation]

ACCURACY AND PRECISION

Table 1.2 : Different between accuracy and precision

Accuracy	Precision
Degree of closeness between the mean of observations and the true values	Degree of closeness of observation values. The closer the values the higher the precision of the observation
How closely a measurement or Observation comes to measuring a true value.	Degree of refinement/consistency Of a group of observations, and is evaluated on Basic of discrepancy size
The absolute nearness of the measured quantity To its true value [smaller difference means high accuracy]	The nearness of the measured quantity to its average/mean
Includes both random & systematic Effects	Includes only random effects

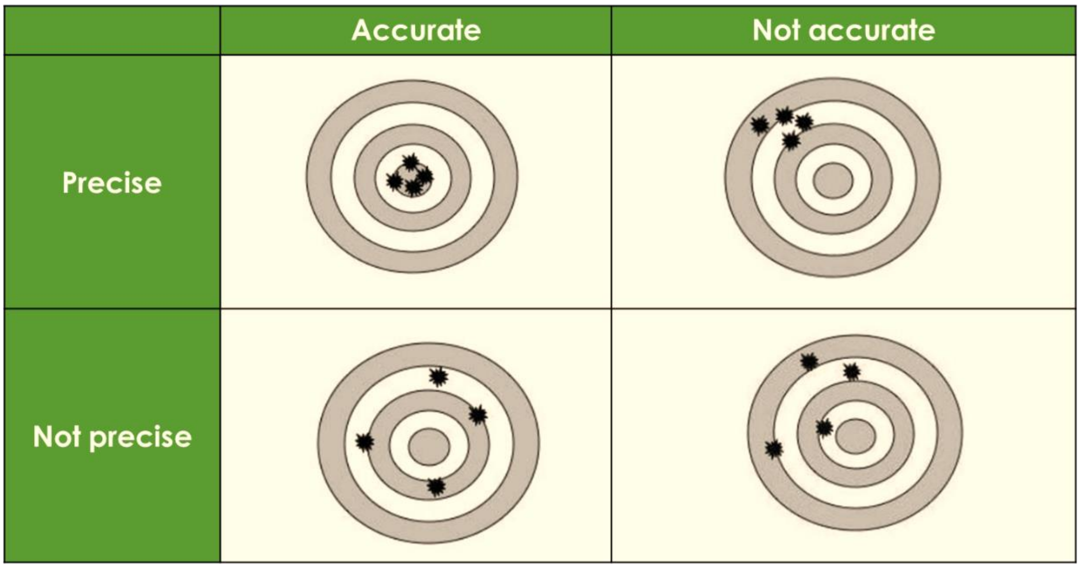


Figure 1.1: Comparison between accuracy and precise

ERROR IN SURVEY MEASUREMENT

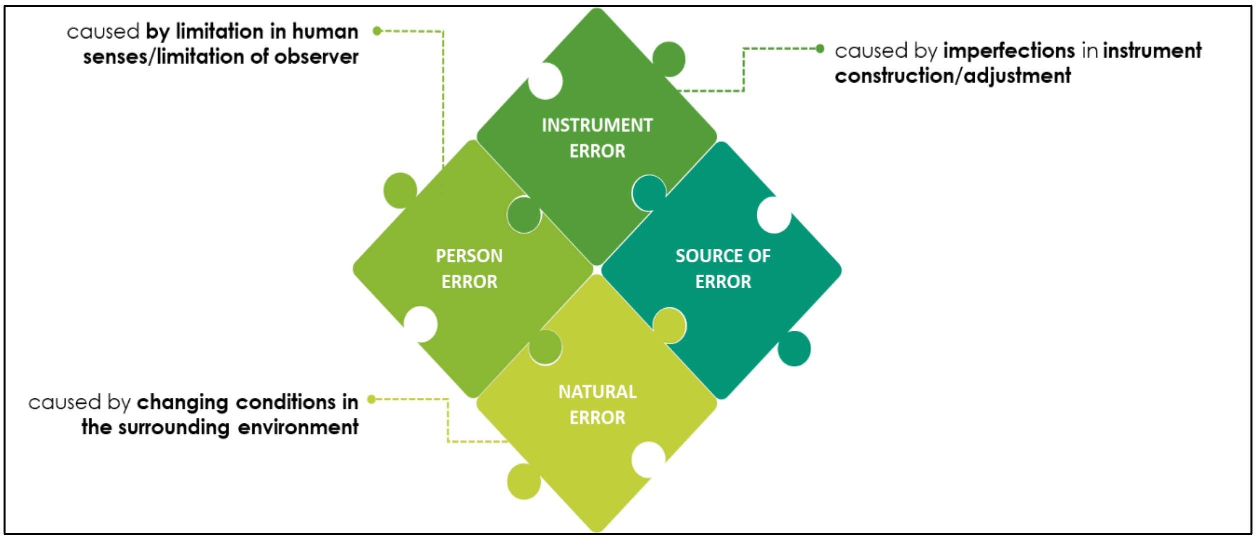


Figure 1.2 : Source of error in Survey Observation

TYPES OF ERROR

Gross error

- The result of blunders or mistakes that are due to carelessness of the observer.
- They are not classified as errors and must be removed from any set of observations.

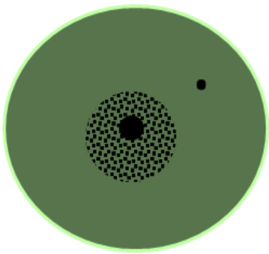
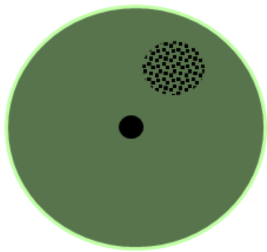
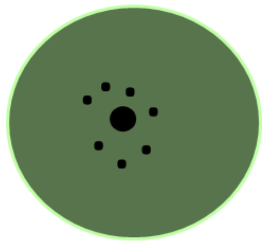
Systematic error

- These errors follow some physical law and thus these errors can be predicted.
- Some systematic errors are removed by following correct measurement

Random error

- These are the errors that remain after all mistakes and systematic errors have been removed from the measured value.
- the result of human and instrument imperfections.

Table 1.3 : Types of error

Gross Error	Systematic Error	Random Error
caused by confusion or by an observer's carelessness.	• Biases • Factor more to measuring system	**Remain** in measurement **after gross** and **systematic errors** have been **eliminated**.
		
They are not classified as errors and must be removed from any set of observations	**If condition change**, **magnitude of systematic errors also change.**	-normal distribution table -near to 0 in positive or negative value
• **Solution**: **must be detected & eliminated**	Solution: calibration, model the error	Solution: LSA

TUTORIAL

1. Determine purpose of survey adjustment

2. Justify advantages of Survey adjustment

3. Describe the term of accuracy and precision in land survey.

4. Explain the types of error in measurement
   i. Gross error
   ii. Systematic error
   iii. Random error

5. Sketch a suitable diagram and state the meaning of accuracy and precision

6. Explain the source of error in measurement
   i. Instrument error
   ii. Natural error
   iii. Personal error

7. Explain two types of Mathematic Model
   i. Functional Model
   ii. Stochastic Model

Chapter 2: Statistic & Analysis

Numerical Statistical Sample

Mean

Mean is the average of the observation.

$$\bar{x} = \frac{\sum x}{n}$$

$$\bar{x} = mean$$

$$\sum x = total\ of\ observation$$

$$n = number\ of\ observation$$

Mode

most commonly observed value in a set of data

Median

- The midpoint of sample data set when arranged in ascending or descending order.
- If number of sample data is even, the average of the two observations at middle data set is used to reprehend as median

Range

Range is the different between the highest and lowest value. It provides an indication of the precision of the data

range = max value of observation – min value of observation

Middle range

The middle range or middle extreme is a measure of central tendency of a sample data defined as the arithmetic mean of the maximum and minimum values of the data set.

$$\boldsymbol{mid\ range = \frac{\max value\ of\ observation + \min value\ of\ observation}{2}}$$

Example 1

An EDM instrument and reflector are set at the ends of a baseline. Its length is measured 9 times with the following results. Calculate mean, median, mode, range.

60.214	60.217	60.214
60.215	60.211	60.219
60.214	60.213	60.212

Answer

Table 2.1 : rearrange data in ascending odder

Observation	Height
1	60.211
2	60.212
3	60.213
4	**60.214**
5	**60.214**
6	**60.214**
7	60.215
8	60.217
9	60.219
TOTAL	**541.929**

Middle Point

$$mean = \frac{\sum x}{n} = \frac{481.718}{9} = 60.214$$

$$mode = 60.214$$

$$median = 60.214$$

$$range = 60.211 - 60.219 = 0.008$$

Example 2

Base on table 2.1, calculate mean, median, mode, range & middle range

Table 2.2 : Observation data

Observation	Height
1	35.421
2	35.432
3	35.425
4	35.423
5	35.425
6	35.421
7	35.425
8	35.430
9	35.420
10	35.419

Answer

Mean = $\frac{345.241}{10} = 34.424$

Range = 35.432 – 35.419 = 0.013

Middle range = $\frac{35.432+35.419}{2} = 35.4255$

Median= $\frac{35.432+35.425}{2} = 35.424$

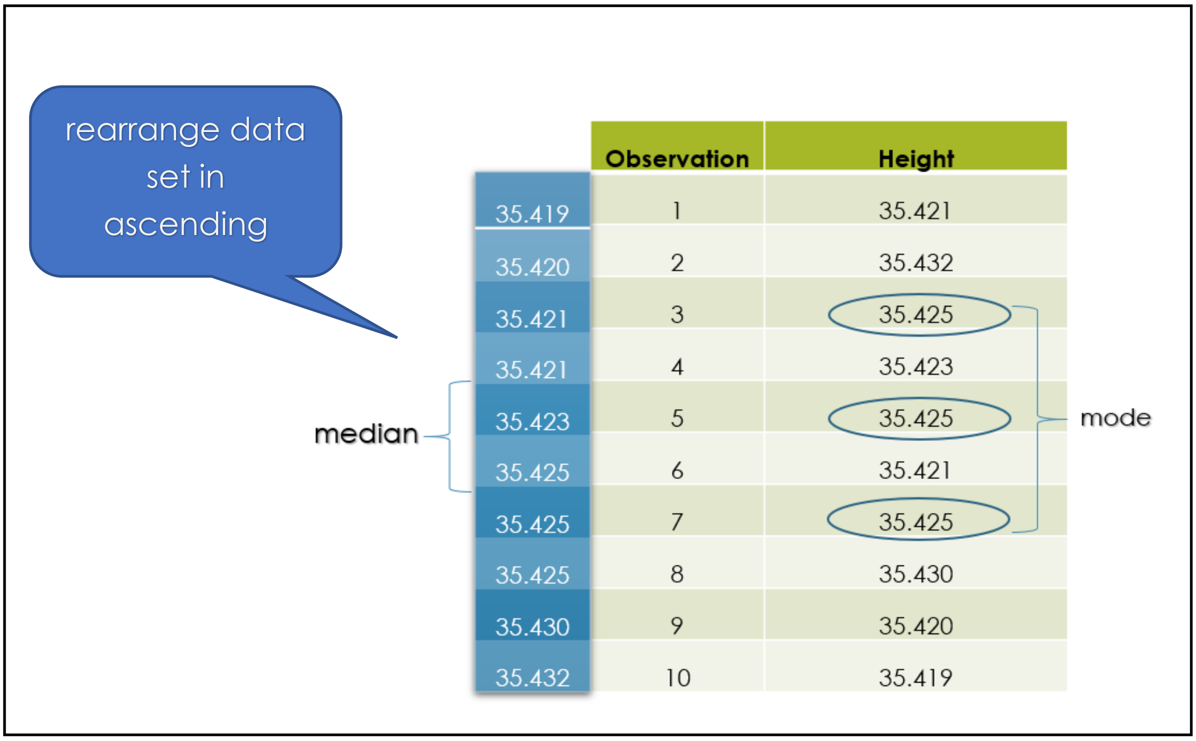


Figure 2.1 : Step to get median and mode

Table 2.3 : Answer for this question

ITEM	value
mean	35.424
mode	35.425
median	35.424
range	0.013
middle range	35.4255

Population

Population consists of all possible measurement that can be made on a particular item or procedure. Often, a population has an infinite number of data element.

Sample

Sample is a subset of data selected from the population.

True value

A quantity's theoretically correct or exact value

The true value is simply the population's arithmetic mean if all repeated observations have equal precision.

1. No measurement is exact
2. Every measurement contains errors
3. The true value of measurement is never known
4. The exact size of the error present is always unknown

Error Propagation

Error Propagation is the distribution of error

Most probable value

The most probable value is that value for a measured quantity which based on the observation, has the highest probability of occurrence.

Error

- The difference between a measured value for any quantity and its true value.
- Error exists in all observation

$$\boldsymbol{\varepsilon = y - \mu}$$

$$\varepsilon = the\ error\ in\ a\ observation$$

$$y = the\ measured\ value$$

$$\mu = true\ value$$

Residual

- A residual is the difference between any individual measured quantity and the most probable value for that quantity.
- Residuals are the values that are used in adjustment computations since most probable values can be determined.
- The term *error* is frequently used when *residual* is meant, and although they are very similar and behave in the same manner, there is this theoretical distinction.
- Residual = computed value [or mean] - observed value

$$\boldsymbol{v = \bar{x} - x}$$

$$v = the\ residual\ in\ the\ observation$$

$$\bar{x} = most\ probable\ value\ for\ the\ unknown$$

$$x = individual\ observation$$

Degree of freedom or redundancies

- The degrees of freedom are the number of observations that are in excess of the number necessary to solve for the unknowns.
- The number of degrees of freedom equals the number of *redundant* observations

Variance

- This is a value by which the precision is given for a set of data.
- The mean of the square of the errors

$$\boldsymbol{S^2 = \frac{\sum(x-\overline{x})^2}{n-1}}$$

$$\boldsymbol{S}^2 = variance$$

$$x = observation\ data$$

$$\bar{x} = mean\ od\ data\ set$$

$$n = number\ of\ observation$$

Standard error

The square root of the population variance

Standard variation

- The square root of the sample variance
- Small std. dev = good data/good observation

  Small std. dev = small changing/ a bit movement of structure/land slide

$$\boldsymbol{s = \sqrt{s^2}}$$

$$\boldsymbol{s} = std.deviation$$

$$\boldsymbol{S}^2 = variance$$

Standard variance of mean

The mean is computed from the sample standard deviation

Covariance

- Covariance is Correlation between the two unknow variable.
- If the covariance value decreases, the Correlation of the variable also decreases.
- Correlation coefficient and Covariance give an indication of the relationship between variables.

$$\sigma_{xy} = \frac{\sum(\boldsymbol{x} - \overline{\boldsymbol{x}})(\boldsymbol{y} - \overline{\boldsymbol{y}})}{\boldsymbol{n} - \mathbf{1}}$$

$$\sigma_{xy} = covariance$$

$$x = observation\ data\ for\ 1st\ variable$$

$$\bar{x} = mean\ for\ data\ set\ x$$

$$y = observation\ data\ for\ 2nd\ variable$$

$$\bar{y} = mean\ for\ data\ set\ x$$

$$n\ = number\ of\ observation$$

Correlation coefficient

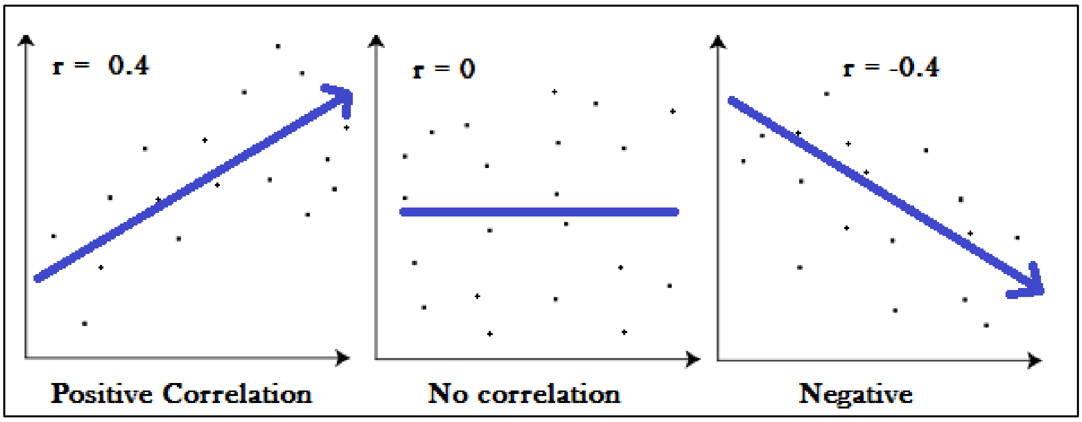


Figure 2.2 : Correlation coefficient must between 1 to -1 only

- **A correlation coefficient of 1** means that amount for variable A increase in (almost) perfect correlation with variable B
- **A correlation coefficient of -1** means that the amount of variable A decrease in (almost) perfect correlation with variable B
- **Zero** means that no correlation between two variables.

$$\rho_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 . \sum(y-\bar{y})^2}} ; \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$

$$\rho_{xy} = correlation\ coefficient$$

$$\sigma_{xy} = covariance$$

$$x = observation\ data\ for\ 1st\ variable$$

$$\bar{x}\ \&\ \bar{y} = mean\ for\ data\ set$$

$$y = observation\ data\ for\ 2nd\ variable$$

Table 2.4: Correlation coefficient data analysis

Correlation value	Result
0	Completely uncorrelated
1	Completely positively correlated
-1	Completely negative correlated
$0 < \rho_{xy} < 0.35$	Weak correlation
$0.35 < \rho_{xy} < 0.75$	Significant correlation
$0.75 < \rho_{xy} < 1$	Strong correlation

<table>
<tr><th colspan="3">Statistic Formula</th></tr>
<tr><td>Variance :
$S^2 = \frac{\sum(x-\bar{x})^2}{n-1}$</td><td>Standard Deviation:
$s = \pm\sqrt{variance, s^2}$</td><td>Correlation Coefficient :
$\rho_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 . \sum(y-\bar{y})^2}}$
$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$</td></tr>
<tr><td>Variance for arithmetic mean:
$= \frac{S^2}{n}$</td><td>Std. dev for arithmetic mean:
$= \frac{S}{\sqrt{n}}$</td><td>covarience:
$\sigma_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{n-1}$</td></tr>
<tr><td colspan="3">S^2=variance for sample   x= data $\bar{x}$=mean
$\sigma^2 = variance\ for\ population$</td></tr>
</table>

Figure 2.3: statistic formula

Example 3

Find variance,s^2 and standard deviations for height observation. Find correlation coefficient, ρ_{xy} between height and volume covariance.

Table 2.5 : Observation for height and volume

Observation	Height, x	Volume, y
1	35.421	5313
2	32.552	4883
3	33.210	4982
4	33.213	4982
5	30.441	4566
6	29.554	4433
7	35.487	5323
8	36.481	5472
9	35.420	5313
10	36.221	5433

NOTE :

For this question, create table like table 2.4

Follow the formula to solve this question

STEP 1 : Create table

Table 2.6 : Answer sheet

Obs	Height, x	Volume, y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x}).(y-\bar{y})$
1	35.421	5313	1.621	243	2.628	59049	393.9
2	32.552	4883	-1.248	-187	1.558	34969	233.38
3	33.210	4982	-0.59	-88	0.348	7744	51.92
4	33.213	4982	-0.587	-88	0.345	7744	51.656
5	30.441	4566	-3.359	-504	11.283	254016	1692.9
6	29.554	4433	-4.246	-637	18.029	405769	2704.7
7	35.487	5323	1.687	253	2.846	64009	426.81
8	36.481	5472	2.681	402	7.188	161604	1077.8
9	35.420	5313	1.62	243	2.624	59049	393.66
10	36.221	5433	2.421	363	5.861	131769	878.82
TOTAL	**338.000**	**50700.000**			**52.709**	**1185722**	**7905.549**

Step 3 : Calculate mean for sample, $\bar{x}$ and $\bar{y}$

$$mean, \bar{x} = \frac{\sum x}{n}$$

$$= \frac{338}{10}$$

= 33.8

$$mean, \bar{y} = \frac{\sum y}{n}$$

$$= \frac{50700}{10}$$

= 5070

Step 3 : Calculate variance for sample, s^2

$$Variance\ x,\ S^2 = \frac{\sum(x-\bar{x})^2}{n-1}$$

$$= \frac{52.709}{9}$$

= 5.856509

$$Variance\ y,\ S^2 = \frac{\sum(y-\bar{y})^2}{n-1}$$

$$= \frac{1185722}{9}$$

= 131746.889

Step 4 : Calculate standard deviation for sample, s

$$\boldsymbol{std.dev\ x},\ \boldsymbol{s} = \pm\sqrt{variance, s^2}$$

$$= \sqrt{5.856509}$$

$$= 2.420023$$

$$\boldsymbol{std.dev\ y},\ \boldsymbol{s} = \pm\sqrt{variance, s^2}$$

$$= \sqrt{131746.889}$$

$$= 362.970$$

Step 5 : Calculate covariance for sample, $\boldsymbol{\sigma_{xy}}$

$$\boldsymbol{Covariance}\ \sigma_{xy} = \frac{\sum(\boldsymbol{x}-\overline{\boldsymbol{x}})(\boldsymbol{y}-\overline{\boldsymbol{y}})}{\boldsymbol{n-1}}$$

$$= \frac{7905.549}{9}$$

$$= 878.394$$

Step 6 : Calculate correlation coefficient, $\boldsymbol{\rho_{xy}}$

$$\boldsymbol{\rho_{xy}} = \frac{\sum(\boldsymbol{x}-\overline{\boldsymbol{x}})(\boldsymbol{y}-\overline{\boldsymbol{y}})}{\sqrt{\sum(\boldsymbol{x}-\overline{\boldsymbol{x}})^2 . \sum(\boldsymbol{y}-\overline{\boldsymbol{y}})^2}}$$

$$= \frac{7905.549}{\sqrt{(52.709)*(1185722)}}$$

$$= 0.99$$

Example 4

Table 2.7 show the data obtained from distance measurement work. Calculate mean, mean error, variance and standard deviation.

Table 2.7 : Observation table

Observation	Distance, x
1	35.421
2	32.552
3	33.210
4	33.213
5	30.441
6	29.554

Answer

STEP 1 : Create table

Table 2.8 : Answer sheet

Observation	Distance, x	Mean error $(x-\bar{x})$	$(x-\bar{x})^2$
1	40.209	-0.003	0.000009
2	40.207	-0.005	0.000025
3	40.211	-0.001	0.000001
4	40.219	0.007	0.000049
5	40.206	-0.006	0.000036
6	40.218	0.006	0.000036
TOTAL	**241.270**		**0.000156**

Step 2 : Calculate mean for sample, $\bar{x}$

$$mean, \bar{x} = \frac{\sum x}{n}$$

$$= \frac{241.270}{6}$$

= 33.8

Step 3 : Calculate variance for sample, s^2

$$Variance\ x,\ S^2 = \frac{\sum(x - \bar{x})^2}{n - 1}$$

$$= \frac{0.000156}{6}$$

$$= 2.6 \times 10^{-5}$$

Step 4 : Calculate standard deviation for sample, s

$$\boldsymbol{std.dev\ x,\ s} = \pm\sqrt{variance, s^2}$$

$$= \sqrt{2.6 \times 10^{-5}}$$

= 0.005

Example 5

The given data are: -Calculate standard deviation and covariance

$$variance, \sigma_x^2 = 0.3035\ cm^2$$

$$variance, \sigma_y^2 = 0..5421\ cm^2$$

Correlation coefficient, $\rho_{xy} = 0.892$

Standard deviation x. $\sigma_x = \sqrt{\sigma_x^2}$

$= \sqrt{\ 0.3035}$

= 0.5509

Standard deviation y. $\sigma_y = \sqrt{\sigma_y^2}$

$= \sqrt{\ 0.5421}$

= 0.7362744

Note :

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x . \sigma_y}$$

Covariance, $\sigma_{xy} = \rho_{xy} \times \sigma_x . \sigma_y$

$= 0.892 \times 0.635216 \times 0.7362744$

= 0.4171824051

Tutorial

1. Describe the term of
   i. Mean
   ii. Median
   iii. Mode
   iv. Range
   v. Middle range

2. Describe the term of
i. Variance
ii. Standard deviation

3. Explain different between error and residual

4. Table 2.8 shows the data obtained from angle measurement work. Calculate mean, mean error, variance and standard deviation.

Table 2.8: Observation data

Observation	Angle
1	$60° \; 20'15''$
2	$60° \; 20'20''$
3	$60° \; 19'55''$
4	$60° \; 20'25''$
5	$60° \; 20'30''$
6	$60° \; 19'50''$

5. From the numerical data set, calculate mean, mode, and variance

50.412	50.400	50.421
50.412	50.420	50.419
50.415	50.417	50.412

6. Table 2.9 shows the data obtained from angle measurement work. Calculate variance, standard deviation, covariance and correlation coefficient.

Table 2.9: Observation data

Obs.	Distance (X) meter	Distance (Y) meter
1	39.110	48.550
2	39.020	48.700
3	39.680	48.900
4	39.450	48.880
5	39.770	48.654

Chapter 3: Variance-Covariance Propagation

Variance

- This is a value by which the precision is given for a set of data.
- The mean of the square of the errors.

Covariance

- Covariance is coloration between the two unknow variable.
- If the covariance value decreases, the coloration of the variable also decreases

Properties of variance-covariance matrix

1. Symmetric matrix
2. Determinant of covariance matrix should not equal to zero
3. All diagonal element in covariance matrix must positive

covariance

$$\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x1x2} \\ \sigma_{x2x1} & \sigma_{x_2}^2 \end{bmatrix}$$

variance

Example 1

$$\sigma_y^2 = \begin{bmatrix} \mathbf{70} & \mathbf{23.4} \\ \mathbf{23.4} & \mathbf{36.5} \end{bmatrix}$$

$$variance, \sigma = y_1 = 70\ ;\ y_2 = 36.5$$

$$covariance = 23.4$$

Table 3.1: Properties of variance-covariance matrix

Symmetric matrix	Diagonal element positive
$\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ matrix A= matrix $\boldsymbol{A^T}$	$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ All diagonal element +ve
$\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} \end{bmatrix}$ Not symmetric	$\boldsymbol{A} = \begin{bmatrix} \mathbf{-4} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ One of diagonal element negative

LOPOV – Law of Propagation of Variance (Error)

$$y = x_1 + x_2 + c_1$$

$$\Downarrow$$

$$y \pm \sigma_y = (x_1 \pm \sigma_{x1}) + (x_1 \pm \sigma_{x1})$$

Variance- covariance

$$\sigma_y^2 = A\,\sigma_x^2 A^T$$

In matric form, if the n unknow are indepandance, so covariance element in matrix is zero.

$$\sigma_y^2 = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_1} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x1x2} \\ \sigma_{x2x1} & \sigma_{x_2}^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_1} \end{bmatrix}$$

For nonlinear problem

$$\sigma_y^2 = \left(\frac{\partial y}{\partial x_1} . \sigma_{x_1}\right)^2 + \left(\frac{\partial y}{\partial x_2} . \sigma_{x_2}\right)^2$$

STEP BY STEP FOR NONLINEAR PROBLEMS

1. Identify the suitable formula
2. Differentiation each element from formula
3. Substitute into nonlinear equation
4. Find estimated Standard deviation /estimated error

Differentiation notes

Equation	1st derivative
$y = ax$	$\frac{\partial y}{\partial x} = a$
$y = ax^n$	$\frac{\partial y}{\partial x} = n.a.x^{n-1}$
$y = ax^n + bx$	$\frac{\partial y}{\partial x} = n.a.x^{n-1} + b$
$y = sin\theta$	$\frac{\partial y}{\partial x} = cos\theta$
$y = cos\theta$	$\frac{\partial y}{\partial x} = -sin\theta$
$y = a\,sin\theta$	$\frac{\partial y}{\partial x} = acos\theta$

Example 2: Differentiation notes

1. Find first derivative for y respect to b

$$y = 2b^3 + Cb$$

$$\frac{dy}{db} = 3 \times 2b^{3-1} + C$$

$$= 6b^2 + C$$

2. Find first derivative for D respect to β

$$D = 23.10 \cos\beta$$

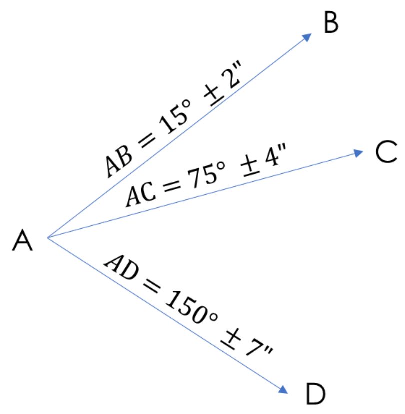
$$\frac{dD}{d\beta} = 23.10\ (-sin\beta)$$

$$= -23.10\ sin\beta$$

Example 3 : Matric form

Bering observation data for $AB = 15° \ \pm 2''$; $AC = 75° \ \pm 4''$; $AD = 150° \ \pm 7''$.

Calculate value for angle BAC and CAD, standard deviation and Correlation.



Step 1 : Model the equation observation

BAC = CA – BA

CAD = DA - CA

Step 2 : Apply LOPOV

$$\sigma_y^2 \ = A\, \sigma_x^2 A^T$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 4^2 & 0 \\ 0 & 0 & 7^2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 16 & 0 \\ 0 & -16 & 49 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 & -16 \\ -16 & 65 \end{bmatrix}$$

$std.dev_{BAC} = \sqrt{20}$; $\qquad std.dev_{CAD} = \sqrt{65}$.

$$correlation = \frac{-16}{\sqrt{20 \times 65}}$$

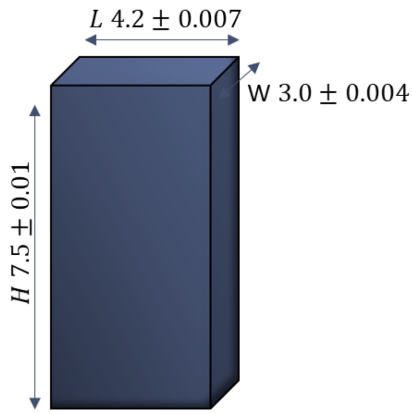
$Angle\ BAC \ = \ 60° \ \pm 4.47''$

$Angle\ BAC \ = \ 75° \ \pm 8.06''$

Example 4 : Nonlinear problem

The dimensions of a rectangular tank are measured. Calculate the tank's volume and its estimated standard deviation using the measurements above.

Dimension, meter	Std. dev, meter
H = 7.500	± 0.010
W = 3.000	± 0.007
L = 4.200	± 0.004



Step 1 : Identify the suitable formula

$$Volume\ rectangular, V = LWH$$

$$V = 7.5 \times 3.0x \times 4.2$$

$$V = 94.500\ meter^3$$

Step 2 : Differentiation each element from formula

$$V = LWH$$

Derivative of V with respect to L

$$\frac{\partial V}{\partial L} = WH = 3.0 \times 7.5 = 22.5\ m$$

Derivative of V with respect to W

$$\frac{\partial V}{\partial W} = LH = 4.2 \times 7.5 = 31.5\ m$$

Derivative of V with respect to H

$$\frac{\partial V}{\partial H} = LW = 4.2 \times 3.0 = 12.6\ m$$

Note:

Find 1st derivative of V with respect to L, W and H

Step 3 : Substitute into nonlinear equation

$$S_v = \sqrt{\left(\frac{\partial V}{\partial L} \cdot S_L\right)^2 + \left(\frac{\partial V}{\partial W} \cdot S_W\right)^2 + \left(\frac{\partial V}{\partial H} \cdot S_H\right)^2}$$

$$= \sqrt{(22.5 \times 0.004)^2 + (31.5 \times 0.007)^2 + (12.6 \times 0.010)^2}$$

$$= \sqrt{0.09^2 + 0.2205^2 + 0.126^2}$$

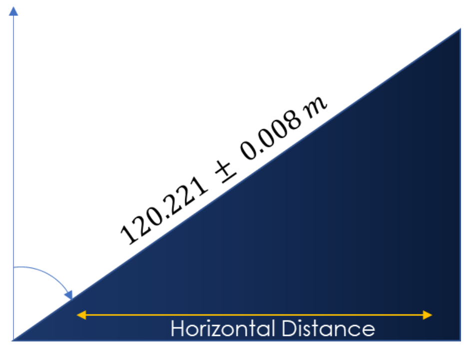
=0.269 m

Step 4 : Standard deviation for volume

$$V = 94.500\ meter^3 \ \pm 0.269$$

Example 6:

A slope distance is observed as $120.221 \pm 0.008\,m$. the vertical angle is observed as 88° 40′10 ± 8.8". what are the horizontal distance and its estimated error.



$V_A = 88°40'100"$

Step 1 : Identify the suitable formula

$$Horizontal\ angle, H_D = S_D \sin\theta$$

$$H_D = 120.221 \sin 88°\ 40'10"$$

$$H_D = 120.189\ m$$

Step 2 : Differentiation each element from formula

$$H_D = S_D \sin\theta$$

Derivative of H_D with respect to S_D

$$\frac{\partial H_D}{\partial S_D} = \sin\theta$$

= sin 88° 40′10" = 0.99973

Derivative of H_D with respect to θ

$$\frac{\partial H_D}{\partial \theta} = S\,(\cos\theta)$$

= 120.221 cos 88° 40′10"

= 2.7916

Note:

Find 1st derivative of H_D with respect to $S_D\ and\ \theta$

Step 3 : Substitute into nonlinear equation

$$S_{H_D} = \sqrt{\left(\frac{\partial H_D}{\partial S_D} \cdot S_{S_d}\right)^2 + \left(\frac{\partial H_D}{\partial \theta} \cdot S_\theta\right)^2}$$

$$= \sqrt{(0.99973 \ \cdot 0.008)^2 \ + \left(2.7916 \ \cdot \ \left(8.8'' \times \frac{\pi}{180}\right)\right)^2}$$

= 0.008 m

Change degree to radians

Step 4 : Standard deviation for horizontal distance

$$H_D = 120.189 \ m \pm 0.008$$

Example 7:

The radius of a given tank is 13.00m ± 0.003m. Its height is 26.00m ± 0.006m. The mathematical model for the tank volume is $V = \pi r^2 h$. Calculate

i. Volume of Standard tank

ii. Std. deviation of the volume

Step 1 : Identify the suitable formula

$$Volume, V = \pi r^2 h$$

$$= \pi(13)^2 26$$

$$= 13804.158\, m^3$$

Step 2 : Differentiation each element from formula

$$V = \pi r^2 h$$

Derivative of V with respect to r

$$\frac{dv}{dr} = 2\pi rh = 2\pi\,(13)(26) = 2123.717$$

Derivative of V with respect to h

$$\frac{dv}{dh} = \pi r^2 = \pi(13)^2 = \ 530.929$$

Note:

Find 1st derivative of V with respect to r and h

Step 3 : Substitute into nonlinear equation

$$S_V = \sqrt{\left(\frac{dv}{dr}.s_r\right)^2 + \left(\frac{dv}{dh}.s_h\right)^2}$$

$$S_V = \sqrt{(2123.717\ \times 0.013)^2 + (530.929\ \times 0.026)^2}$$

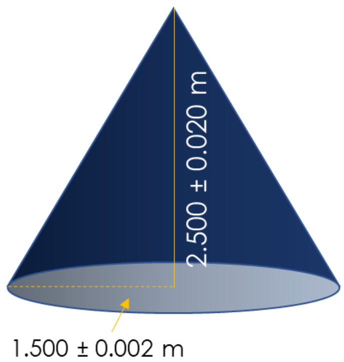
$$= 30.867$$

Step 4 : Standard deviation for volume

$$V = 13804.158\, m^3 \pm 30.867$$

Example 8

The measured height of the cone is 2.500 ± 0.020 m. The measured radius is 1.500 ± 0.002 m. Calculate the variance covariance propagation of the volume.



Step 1 : Identify the suitable formula

$$Volume, V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(1.5)^2 2.5$$

$$= 5.890\ m^3$$

Step 2 : Differentiation each element from formula

$$V = \frac{1}{3}\pi r^2 h$$

Note:

Find 1st derivative of V with respect to r and h

Derivative of V with respect to r

$$\frac{dV}{dr} = \frac{2}{3}\pi rh = \frac{2}{3}\pi(1.5)(2.5) = 7.854\ m$$

Derivative of V with respect to h

$$\frac{dV}{dh} = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi(1.5)^2 = 2.356\ m$$

Step 3 : Substitute into nonlinear equation

$$\sigma_V = \sqrt{\left(\frac{dV}{dr}.\sigma_r\right)^2 + \left(\frac{dV}{dh}.\sigma_h\right)^2}$$

$$= \sqrt{(7.854 \times 0.002)^2 + (2.356 \times 0.020)^2}$$

$$= 0.050$$

Step 4 : Standard deviation for volume

$$V = 5.890\ m^3 \pm 0.05$$

TUTORIAL

1. A slope distance is observed as $60.752 \pm 0.008\ m$. The vertical angle is observed as $87° 23'10 \pm 6.5"$. What are the horizontal distance and its estimated error.

2. A horizontal distance is observed as $30.455 \pm 0.008\ m$ from the building A. The vertical angle is observed as $71° 14'20 \pm 8.8"$. What are the height of building of and its estimated error.

3. A storage tank in the shape of cylinder has a measured height of $12.2 \pm 0.023\ m$ and a radius of $2.3 \pm 0.005\ m$. What are the tank's volume and estimated error in this volume.

4. A rectangular container has dimensions of $5.5 \pm 0.004\ m$ by $7.45 \pm 0.005\ m$. What is the area of the parcel and the estimated area in this area.

"You can't claim you tried everything if you never got up in the last third of the night to ask Allah for it"

Chapter 4: Weight of Observation

Weight Of Observation

- A measure of an observation's worth compared to other observations.
- Weight is a positive number assigned to an observation that indicates the relative accuracy to other observations
- Weight ae used to control the sizes of corrections applied to observation in an adjustment.
- The more precise an observation, the higher its weight
- The smaller the variance, the higher the weight.
- With uncorrelated observations, weights of the observations are inversely proportional to their variances.

$$w = \frac{1}{\sigma^2}$$

$$w = weight$$

$$\sigma^2 = variance$$

For levelling

$$w = \frac{1}{d}$$

$$w = weight$$

$$d = distance$$

Weighted Mean

A mean value computed from weighted observations. Weighted mean is the most probable value for a set of weighted observation.

$$\bar{z} = \frac{\sum x(w)}{\sum w}$$

$$\bar{z} = weighted\ mean$$

$$x = observation\ data$$

$$w = weight$$

Standard deviation of weighted mean

$$S_{\bar{z}} = \sqrt{\frac{\sum wv^2}{(\sum w)(n-1)}}$$

$$S_{\bar{z}} = std.dev\ of\ weighted\ mean$$

$$w = weight$$

$$v = residual$$

$$n = number\ of\ observation$$

Std. dev of weighted for observation

$$S_n = \sqrt{\frac{\sum wv^2}{w_n(n-1)}}$$

$$S_n = std.dev\ of\ weighted\ for\ observation$$

$$w = weight$$

$$v = residual\ (mean, \bar{z} - observation, x)$$

$$n = number\ of\ observation$$

Std. dev of weighted unit

$$S_w = \sqrt{\frac{\sum wv^2}{(n-1)}}$$

$$S_w = std.\,dev\ of\ weighted\ unit$$

$$w = weight$$

$$v = residual$$

$$n = number\ of\ observation$$

Example 1

Data for distance is observed using three different types of instrument. Calculate:-

i. Weight mean

ii. Std. dev of weighted mean

iii. Std. dev of weighted observation

iv. Std. dev of weighted unit

Instrument	Distance AB	Weight
EDM	15.231	3
Disto meter	15.235	2
Tape	15.220	1

Step 1: Calculate Weight mean

$$weight\ mean, \bar{z} = \frac{\sum x(w)}{\sum w}$$

$$\bar{z} = \frac{15.231(3) + 15.235(2) + 15.220(1)}{3 + 2 + 1} = 15.2305$$

Step 2: Create table

Instrument	Distance AB, x	Weight	$v = \bar{z} - x$	$w.v^2$
EDM	15.231 m	3	0.0005	7.5 x10^{-7}
Disto meter	15.235 m	2	0.0045	4.05 x10^{-5}
Tape	15.220 m	1	0.0105	1.1x10^{-4}
Total		6		0.00015

Step 3: Calculate std. dev of weighted mean

$$std.dev\ of\ weighted\ mean, S_{\bar{z}} = \sqrt{\frac{\sum wv^2}{(\sum w)(n-1)}}$$

$$S_{\bar{z}} = \sqrt{\frac{0.00015}{(6)(2)}} = 0.0035\ m$$

Step 3: Calculate std. dev of weighted observation

$$std.dev\ of\ weighted\ observation, S_n = \sqrt{\frac{\sum wv^2}{w_n(n-1)}}$$

Weight for each observation

Std deviation of weighted observation distance using EDM

$$S_{EDM} = \sqrt{\frac{0.00015}{3(2)}} = 0.005\ m$$

Std deviation of weighted observation for distance using disto meter

$$S_{DM} = \sqrt{\frac{0.00015}{2(2)}} = 0.006\ m$$

Std deviation of weighted observation for distance using tape

$$S_{TAPE} = \sqrt{\frac{0.00015}{1(2)}} = 0.009\ m$$

Step 4: Calculate std. dev of weight unit

$$std.dev\ of\ \ weight\ unit, S_w = \sqrt{\frac{\sum wv^2}{(n-1)}}$$

$$S_w = \sqrt{\frac{0.00015}{2}} = \ 0.009m$$

Example 2

An angle is observed on three different days with the following results, calculate: -

i. Weight mean

ii. Std. dev of weighted mean

iii. Std. dev of weighted observation

iv. Std. dev of weighted unit

Day	Observation	Weight
1	$30° \, 10'20''$	1
2	$30° \, 10'30''$	3
3	$30° \, 10'50''$	2
4	$30° \, 10'45''$	3
5	$30° \, 10'50''$	4

Step 1 : Calculate weighted mean

$$weight\ mean, \bar{z} = \frac{\sum x(w)}{\sum w}$$

$$\bar{z} = \frac{392° \, 19'05''}{13} = 30° \, 10^{'} 41.92''$$

Step 2 : create table

day	Observation, x	Weight, w	$\boldsymbol{x \times w}$	$\boldsymbol{v = \bar{z} - x}$	$\boldsymbol{w.v^2}$
1	$30° \, 10'20"$	1	$30° \, 10'20"$	$21.92"$	$0.13"$
2	$30° \, 10'30"$	3	$90° \, 31'30"$	$11.92"$	$0.12"$
3	$30° \, 10'50"$	2	$60° \, 21'40"$	$-\, 8.08"$	$0.04"$
4	$30° \, 10'45"$	3	$90° \, 32'15"$	$-3.08"$	$0.01"$
5	$30° \, 10'50"$	4	$120° \, 43'20"$	$-8.08"$	$0.07"$
total		13	$392° \, 19'05"$		$0.37"$

Step 3: Calculate std. dev of weighted mean

$$std.dev\ of\ mean, S_{\bar{z}} = \sqrt{\frac{\sum wv^2}{(\sum w)(n-1)}}$$

$$S_{\bar{z}} = \sqrt{\frac{0.37"}{(13)(4)}} = 5.06"$$

Step 3: Calculate std. dev of weighted observation

$$std.dev\ of\ weighted\ observation\ , S_n = \sqrt{\frac{\sum wv^2}{w_n(n-1)}}$$

Std deviation of weighted observation for 1st day

$$S_1 = \sqrt{\frac{0.37"}{1(4)}} = 18.25"$$

Std deviation of weighted observation for 2nd day

$$S_2 = \sqrt{\frac{0.37"}{3(4)}} = 10.54"$$

Std deviation of weighted observation for 3rd day

$$S_3 = \sqrt{\frac{0.37"}{2(4)}} = 12.9"$$

Std deviation of weighted observation for 4th day

$$S_4 = \sqrt{\frac{0.37"}{3(4)}} = 10.54"$$

Std deviation of weighted observation for 5th day

$$S_5 = \sqrt{\frac{0.37"}{4(4)}} = 9.12"$$

Step 4: Calculate std. dev of weight unit

$$std.\,dev\ of\ weight\ unit, S_w = \sqrt{\frac{\sum wv^2}{(n-1)}}$$

$$S_w = \sqrt{\frac{0.37"}{4}} = 18.25"$$

Example 3

Based on data below, calculate: -

1. Calculate weight mean
2. Std. dev of weighted mean
3. Std. dev of weighted observation
4. Std. dev of weighted unit

BIL	Distance, x_i	$Std.\ dev,\ \sigma_x$
1	30.467	±0.020
2	30.453	±0.014
3	30.448	±0.020
4	30.457	±0.010
5	30.462	±0.010

Step 1 : Calculate weight for each observation

$$weight, w = \frac{1}{\sigma^2}$$

$$w_1 = \frac{1}{0.0004} = 2500$$

$$w_2 = \frac{1}{0.0002} = 5000$$

$$w_3 = \frac{1}{0.0004} = 2500$$

$$w_4 = \frac{1}{0.0001} = 10000$$

$$w_5 = \frac{1}{0.0001} = 10000$$

Step 2: Calculate weight mean

$$weight\ mean, \bar{z} = \frac{\sum x(w)}{\sum w}$$

$$\bar{z} = \frac{913742.5}{30000} = 304.581$$

Step 3 : create table

BIL	Distance, x_i	$Std.\ dev\ \sigma_x$	σ^2	W	$w.x$	v	$v^2.W$
1	30.467 m	±0.020	0.0004	2500	76167.5	-0.0089	0.19802
2	30.453 m	±0.014	0.0002	5000	152265	0.0051	0.13005
3	30.448 m	±0.020	0.0004	2500	76120	0.0101	0.25503
4	30.457 m	±0.010	0.0001	10000	304570	0.0011	0.0121
5	30.462 m	±0.010	0.0001	10000	304620	-0.0039	0.1521
	TOTAL			30000	913742.5	0.0035	0.7473

Step 3: Calculate std. dev of weighted mean

$$std.dev\ of\ weighted\ mean, S_{\bar{z}} = \sqrt{\frac{\sum wv^2}{(\sum w)(n-1)}}$$

$$S_{\bar{z}} = \sqrt{\frac{913742.5}{(30000)(4)}} = 2.759\ m$$

Step 3: Calculate std. dev of weighted observation

$$std.dev\ of\ weighted\ observation, S_n = \sqrt{\frac{\sum wv^2}{w_n(n-1)}}$$

Std deviation of weighted observation for 1st data

$$S_1 = \sqrt{\frac{0.7473}{2500(4)}} = 0.009\ m$$

Std deviation of weighted observation for 2nd data

$$S_2 = \sqrt{\frac{0.7473}{5000(4)}} = 0.006\ m$$

Std deviation of weighted observation for 3rd data

$$S_3 = \sqrt{\frac{0.7473}{2500(4)}} = 0.009\ m$$

Std deviation of weighted observation for 4th data

$$S_4 = \sqrt{\frac{0.7473}{10000(4)}} = 0.004\ m$$

Std deviation of weighted observation for 5th data

$$S_5 = \sqrt{\frac{0.7473}{10000(4)}} = 0.004\ m$$

Step 4: Calculate std. dev of weight unit

$$std.\,dev\ of\ weight\ unit, S_w = \sqrt{\frac{\sum wv^2}{(n-1)}}$$

$$S_w = \sqrt{\frac{0.7473}{4}} = 0.432\,m$$

Tutorial

Question 1

An angle was measured at four different times with the following results. What is the most probable value for the angle and the standard deviation in the mean.

day	Observation, x	Std. dev
1	$120° \, 30'20''$	$\pm \, 6.2''$
2	$120° \, 30'30''$	$\pm \, 9.8''$
3	$120° \, 30'50''$	$\pm \, 5.2''$
4	$120° \, 30'45''$	$\pm \, 4.7''$

Question 2

The distance of the routes and the observed differences in elevations are show below, calculate: -

1. Calculate weight mean
2. Std. dev of weighted mean
3. Std. dev of weighted observation
4. Std. dev of weighted unit

route	Different elevation	distance
1	15.321	120 m
2	15.350	98 m
3	15.334	100 m

Chapter 5: Least Square Adjustment

LEAST SQUARE ADJUSTMENT APPLICATIONS

- A least square adjustment (LSA) is method to estimate or adjust the observations to obtain the most accurate value on these observations using statistical analysis.
- LSE is a systematic & simple method to compute estimated value of variables for unknown quantities from redundant measurements when then number of measurements more than number of variables
- Least-squares adjustment minimizes the sum of the squares of the residuals or weighted residuals.
- Condition of least squares adjustment: number of observations must equal or more than number of variables.
- LSE is not required if no redundant measurements

Step by step to solve LSA problem

1. Model the observation equation
2. Create matrix A, X and L
3. Find matrix $A^T A$
4. Find Determinant for $A^T A$
5. Find minor matrix for $A^T A$
6. Adjoint matrix $A^T A$
7. Inverse matrix $A^T A$
8. Find $A^T L$
9. Solve $x = (A^T A)^{-1} . A^T L$

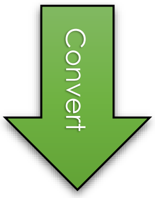
EXAMPLE 1 : Distance

A baseline consists of four stations on a straight-line A, B, C and D are measured using Electronic Distance Measurement device. In order to determine the distance between the stations,

- Determine the number of observation (n) and variables (u).
- By using the matrix method, calculate the adjusted variables for the distance of AB, BC and CD.

Table 4.1: Observation data for baseline

[redacted]	
AB	25.051
BC	25.047
CD	25.110
AC	50.091
BD	50.150
AD	75.200



Note :

1. Identify all the data given
2. Change the data into simple figure

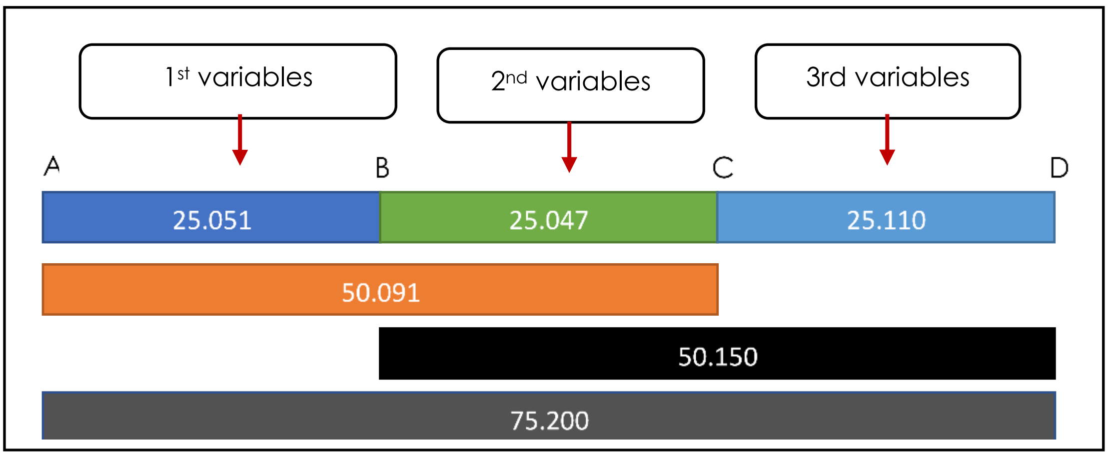


Figure 4.1 : Convert data into simple figure

Number of observation, n = 6

Variables, U = 3   AB , BC & CD

STEP 1 : Model the observation equation

$AB = 25.051 + V_1$

$BC = 25.047 + V_2$

$CD = 25.110 + V_3$

$AB + BC = 50.091 + V_4$

$BC + CD = 50.150 + V_5$

$AB + BC + CD = 75.200 + V_6$

MATRIX

NOTE : Set matrix A base on variables value for each equation

$$A = \begin{array}{c} \begin{array}{ccc} \mathbf{AB} & \mathbf{BC} & \mathbf{CD} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$AB + BC + CD = 75.200 + V_6$

STEP 2 : Create matrix A, X and L

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} ; X = \begin{bmatrix} AB \\ BC \\ CD \end{bmatrix} ; L = \begin{bmatrix} 25.051 \\ 25.047 \\ 25.110 \\ 50.091 \\ 50.150 \\ 75.200 \end{bmatrix}$$

STEP 3 : Find matrix $(\mathbf{A^T A})$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

NOTE : Matrix $\mathrm{A^T}$

Change Row to column

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1+0+0+1+0+1 & 0+0+0+1+0+1 & 0+0+0+0+0+1 \\ 0+0+0+1+0+1 & 0+1+0+1+1+1 & 0+0+0+0+1+1 \\ 0+0+0+0+0+1 & 0+0+0+0+1+1 & 0+0+1+0+1+1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^T A}$)

$$\text{DET } (\boldsymbol{A^T A}) = \begin{bmatrix} 3 & 2 & 1 \\ 2 & \boxed{4} & \boxed{2} \\ 1 & \boxed{2} & \boxed{3} \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 1 \\ \boxed{2} & 4 & \boxed{2} \\ \boxed{1} & 2 & \boxed{3} \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 1 \\ \boxed{2} & \boxed{4} & 2 \\ \boxed{1} & \boxed{2} & 3 \end{bmatrix}$$

$$= 3\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \quad -2\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \quad +1\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

NOTE: Follow this rule

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= 3(12-4) - 2(6-2) + 1(4-4)$$

$$= 3(8) - 2(4) + 1(0)$$

$$= 24 - 8 + 0$$

$$= 16$$

STEP 5: Find minor matrix $A^T A$

$$minor\ (A^T A) = \begin{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{bmatrix}$$

$$minor\ (A^T A) = \begin{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \end{bmatrix}$$

$$minor\ (A^T A) = \begin{bmatrix} 12-4 & 6-2 & 4-4 \\ 6-2 & 9-1 & 6-2 \\ 4-4 & 6-2 & 12-4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

STEP 6 : Find Adjoint matrix ($A^T A$)

$Adj(A^T A) = (cof\ A^T A)^T$

$$cofactor\ (A^T A) = minor(A^T A)\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$cof\ A^T A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \Rightarrow Adj\ A^T A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

Note :

Adjoint matrix

$adj(A^T A) = (cof\ A^T A)^T$

So, for symmetric matrix

$adj(A^T A) = cof\ (A^T A)$

STEP 7: Matrix inverse ($A^T A$)

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \times adj(A^T A)$$

$$= \frac{1}{16} \times \begin{bmatrix} +8 & -4 & +0 \\ -4 & +8 & -4 \\ +0 & -4 & +8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{+8}{16} & \frac{-4}{16} & \frac{0}{16} \\ \frac{-4}{16} & \frac{8}{16} & \frac{-4}{16} \\ \frac{0}{16} & \frac{-4}{16} & \frac{+8}{16} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix}$$

STEP 8: Find A^TL

$$A^TL = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 25.051 \\ 25.047 \\ 25.110 \\ 50.091 \\ 50.150 \\ 75.200 \end{bmatrix}$$

$$A^TL = \begin{bmatrix} 1(25.051) + 0 + 0 + 1(50.091) + 0 + 1(75.200) \\ 0 + 1(25.047) + 0 + 1(50.091) + 1(50.150) + 1(75.200) \\ 0 + 0 + 1(25.110) + 0 + 1(50.150) + 1(75.200) \end{bmatrix}$$

$$A^TL = \begin{bmatrix} 150.342 \\ 200.488 \\ 150.460 \end{bmatrix}$$

STEP 9: Solve $x = (A^TA)^{-1} . A^TL$

$$x = (A^TA)^{-1} . A^TL$$

$$\begin{bmatrix} AB \\ BC \\ CD \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix} \times \begin{bmatrix} 150.342 \\ 200.488 \\ 150.460 \end{bmatrix}$$

$$\begin{bmatrix} AB \\ BC \\ CD \end{bmatrix} = \begin{bmatrix} (0.5)(150.342) + (-0.25)(200.488) + (0)(150.460) \\ (-0.25)(150.342) + (0.5)(200.488) + (-0.25)(150.460) \\ (0)(150.342) + (-0.25)(200.488) + (0.5)(150.460) \end{bmatrix} = \begin{bmatrix} 25.049 \\ 25.0435 \\ 25.108 \end{bmatrix}$$

AB = 25.049 m

BC = 25.0435 m

CD = 25.108 m

Example 2

Between four points A, B, C and D situated on a straight line in pairs distances AB, BC, CD, AC, AD and BD were measured. The six measurements show in table. Calculate the distances of AB, BC and CD by means of linear least squares adjustment.

Table 4.2 : Observation data

line	Distance, m
AB	30.17
BC	10.12
CD	20.25
AC	40.31
AD	60.51
BD	30.36

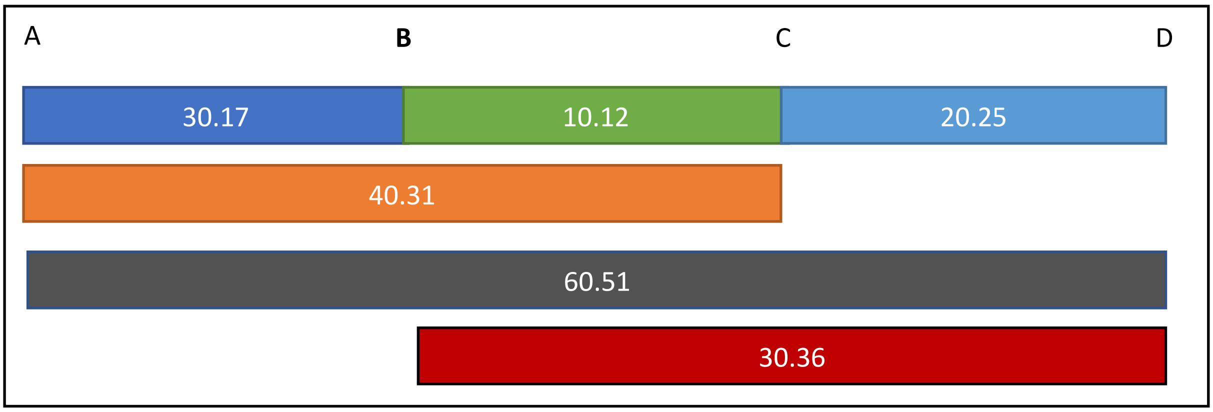


Figure 4.2 : Convert data into simple figure

Step 1: Model the observation equation

$$AB = 30.17 + V_1$$
$$BC = 10.12 + V_2$$
$$CD = 20.25 + V_3$$
$$AB + BC = 40.31 + V_4$$
$$AB + BC + CD = 60.51 + V_5$$
$$BC + CD = 30.36 + V_6$$

STEP 2: Create matrix A, X and L

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \mathrm{X} = \begin{bmatrix} AB \\ BC \\ CD \end{bmatrix}; \mathrm{L} = \begin{bmatrix} 30.17 \\ 10.12 \\ 20.25 \\ 40.31 \\ 60.51 \\ 30.36 \end{bmatrix}$$

STEP 3: Find metric A^TA

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^TA}$)

$$\text{Det } A^TA = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = 3 \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} - 2 \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} + 1 \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 16$$

STEP 5: Minor matrix for (A^TA)

$$\text{Minor } A^TA = \begin{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

STEP 6: Adjoint matrix A^TA

$$Adj(A^TA) = (cof\ A^TA)^T$$

$$cofactor\ (A^TA) = minor(A^TA) \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Cof\ (A^T A) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \quad \longrightarrow \quad Adj\ (A^T A) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

STEP 7: Matrix inverse ($A^T A$)

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \left(adj\ (A^T A) \right)$$

$$= \frac{1}{16} \times \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix}$$

STEP 8: Find $A^T L$

$$A^T L = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 30.17 \\ 10.12 \\ 20.25 \\ 40.31 \\ 60.51 \\ 30.36 \end{bmatrix} = \begin{bmatrix} 130.99 \\ 141.30 \\ 111.12 \end{bmatrix}$$

STEP 9: Solve $x = (A^T A\ )^{-1} . A^T L$

$$x = (A^T A\ )^{-1} . A^T L$$

$$\begin{bmatrix} AB \\ BC \\ CD \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix} \times \begin{bmatrix} 130.99 \\ 141.30 \\ 111.12 \end{bmatrix} = \begin{bmatrix} 30.17 \\ 10.1225 \\ 20.235 \end{bmatrix}$$

AB = 30.170 m

BC = 10.1225 m

CD = 20.235 m

Example 3

EDM instrument is placed at point A and reflector is placed successively at point B, C and D. The observed value AB, AC, AD, BC, CD are show in table. Calculate the unknown value AB, BC and CD

Table 4.3 : Observation data

line	Distance, m
AB	10.231
AC	30.452
AD	52.223
BC	20.225
BD	41.995

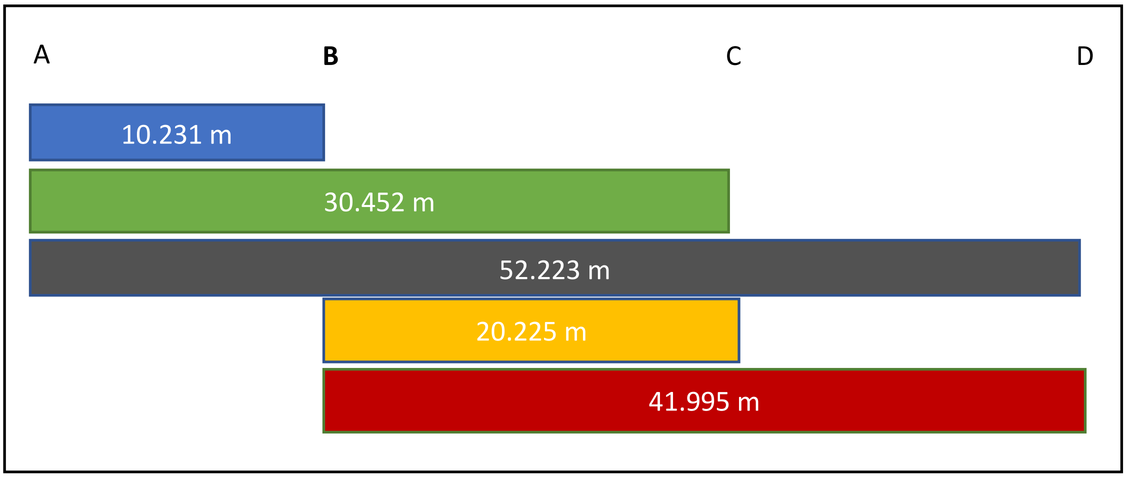


Figure 4.3 : Convert data into simple figure

Step 1: Model the observation equation

$AB = 10.231 + V_1$

$AB + BC = 30.452 + V_2$

$AB + BC + CD = 52.223 + V_3$

$BC = 20.225 + V_4$

$BC + CD = 41.995 + V_5$

STEP 2: Create matrix A, X and L

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \mathrm{X} = \begin{bmatrix} AB \\ BC \\ CD \end{bmatrix}; \mathrm{L} = \begin{bmatrix} 10.231 \\ 30.452 \\ 52.223 \\ 20.225 \\ 41.995 \end{bmatrix}$$

STEP 3: Find metric $A^T A$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathrm{A^T A}$)

$$\text{Det } A^T A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} = 3 \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} - 2 \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} + 1 \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 8$$

STEP 5: Minor matrix for $(A^T A)$

$$\text{Minor } A^T A = \begin{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

STEP 6: Adjoint matrix $A^T A$

$$Adj(A^T A) = (cof\ A^T A)^T$$

$$cofactor\ (A^T A) = minor(A^T A) \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$cof\ (A^T A) = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 8 \end{bmatrix} \Rightarrow adj\ (A^T A) = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

STEP 7: Inverse matrix (A^TA)

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)}\ (\,adj\ (A^TA))$$

$$= \frac{1}{8} \times \begin{bmatrix} 4 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.625 & -0.5 \\ 0 & -0.5 & 1 \end{bmatrix}$$

STEP 8: Find matrix $(\boldsymbol{A^TL})$

$$A^TL = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 10.231 \\ 30.452 \\ 52.223 \\ 20.225 \\ 41.995 \end{bmatrix} = \begin{bmatrix} 92.906 \\ 144.895 \\ 94.218 \end{bmatrix}$$

STEP 9: Solve $\boldsymbol{x = (A^TA\,)^{-1} . A^TL}$

$$x = (A^TA\,)^{-1} . A^TL$$

$$\begin{bmatrix} AB \\ BC \\ CD \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.625 & -0.5 \\ 0 & -0.5 & 1 \end{bmatrix} \times \begin{bmatrix} 92.906 \\ 144.895 \\ 94.218 \end{bmatrix} = \begin{bmatrix} 10.2293 \\ 20.2239 \\ 21.7705 \end{bmatrix}$$

AB = 10.2293 m

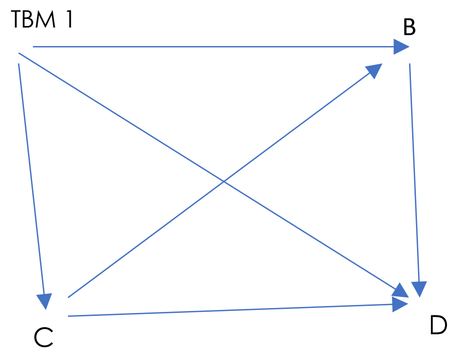
BC = 20.2239 m

CD = 21.7705 m

Example 4 : levelling

Given the height of point TBM 1 is 100.500m. Calculate the adjusted height variable for points B, C and D using Least Square Adjustment observation equation method.

FROM	TO	DIFFERENT HEIGHT
TBM 1	B	0.046
B	D	0.265
TBM 1	D	0.312
TBM 1	C	-0.024
C	B	0.070
C	D	0.336



Step 1: Model the observation equation

$$B - A = 0.046 + V_1$$
$$D - B = 0.265 + V_2$$
$$D - A = 0.312 + V_3$$
$$C - A = -0.024 + V_4$$
$$B - C = 0.070 + V_5$$
$$D - C = 0.336 + V_6$$

Insert known value

$$B - \mathbf{100.5} = 0.046 + V_1$$
$$D - B = 0.265 + V_2$$
$$D - \mathbf{100.5} = 0.312 + V_3$$
$$C - \mathbf{100.5} = -0.024 + V_4$$
$$B - C = 0.070 + V_5$$
$$D - C = 0.336 + V_6$$

New equation

$$B = 100.546 + V_1$$
$$D - B = 0.265 + V_2$$
$$D = 100.812 + V_3$$
$$C = 100.476 + V_4$$
$$B - C = 0.070 + V_5$$
$$D - C = 0.336 + V_6$$

STEP 2: Create matrix A, X and L

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; X = \begin{bmatrix} B \\ C \\ D \end{bmatrix}; L = \begin{bmatrix} 100.546 \\ 0.265 \\ 100.812 \\ 100.467 \\ 0.070 \\ 0.336 \end{bmatrix}$$

STEP 3: Find metric A^TA

$$A^TA = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^TA}$)

$$\text{Det } A^TA = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 3\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - (-1)\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix} + (-1)\begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix} = 16$$

STEP 5: Minor matrix for (A^TA)

$$\text{Minor } A^TA = \begin{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix}$$

STEP 6: Adjoint matrix A^TA

$$Adj(A^TA) = (cof\ A^TA)^T$$

$$cofactor\ (A^TA) = minor(A^TA)\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Cof\ (A^TA) = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{bmatrix}$$

$$adj\ (A^TA) = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{bmatrix}$$

STEP 7: Matrix inverse (A^TA)

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)}\ (\,adj\ (A^TA))$$

$$= \frac{1}{16} \times \begin{bmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

STEP 8: Find A^TL

$$A^TL = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 100.546 \\ 0.265 \\ 100.812 \\ 100.467 \\ 0.070 \\ 0.336 \end{bmatrix} = \begin{bmatrix} 100.351 \\ 100.061 \\ 101.413 \end{bmatrix}$$

STEP 9: Solve $x = (A^TA\,)^{-1} . A^TL$

$$x = (A^TA\,)^{-1} . A^TL$$

$$\begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \times \begin{bmatrix} 100.351 \\ 100.061 \\ 101.413 \end{bmatrix} = \begin{bmatrix} 100.544 \\ 100.4715 \\ 100.8095 \end{bmatrix}$$

B = 100.544 m

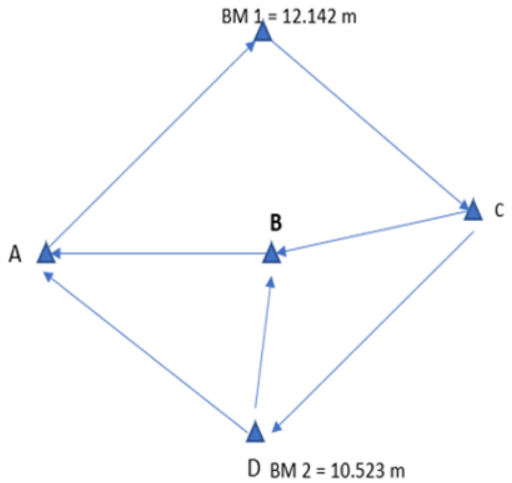
C = 100.472 m

D = 100.810 m

EXAMPLE 5

Calculate the variables for the elevations of A, B and C. Use the least square adjustment method. Given the elevation of BM 1 is 12.142m and BM2=10.523m

FROM	TO	DIFFERENT HEIGHT
A	BM1	-4.425
BM 1	C	2.210
C	B	-2.455
C	BM2	-3.827
BM2	B	1.375
BM2	A	6.040
B	A	4.664



Step 1: Model the observation equation

$BM1 - A = -4.425 + V_1$

$C - BM1 = 2.210 + V_2$

$B - C = -2.455 + V_3$

$BM2 - C = -3.827 + V_4$

$B - BM2 = 1.375 + V_5$

$A - BM2 = 6.040 + V_6$

$A - B = 4.664 + V_7$

NOTE :

- Different height = fore sight - back sight
- Insert know value into the equation

New equation after insert know value

$\mathbf{12.142} - A = -4.425 + V_1$

$C - \mathbf{12.142} = 2.210 + V_2$

$B - C = -2.455 + V_3$

$\mathbf{10.523} - C = -3.827 + V_4$

$B - \mathbf{10.523} = 1.375 + V_5$

$A - \mathbf{10.523} = 6.040 + V_6$

$A - B = 4.664 + V_7$

$A = 16.567 + V_1$

$C = 14.352 + V_2$

$B - C = -2.455 + V_3$

$C = 14.350 + V_4$

$B = 11.898 + V_5$

$A = 16.563 + V_6$

$A - B = 4.664 + V_7$

STEP 2: Create matrix A, X and L

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} X = \begin{bmatrix} A \\ B \\ C \end{bmatrix}; L = \begin{bmatrix} 16.576 \\ 14.352 \\ -2.455 \\ 14.350 \\ 11.898 \\ 16.563 \\ 4.664 \end{bmatrix}$$

STEP 3: Find metric $A^T A$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathrm{A^T A}$)

$$\text{Det } A^T A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = 3 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - (-1) \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix} + 0 \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix} = 21$$

STEP 5: Minor matrix for $(A^T A)$

$$\text{Minor } A^T A = \begin{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 & -3 & 1 \\ -3 & 9 & -3 \\ 1 & -3 & 8 \end{bmatrix}$$

STEP 6: Adjoint matrix $A^T A$

$$Adj(A^T A) = (cof\ A^T A)^T$$

$$cofactor\ (A^T A) = minor(A^T A) \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$cofactor\ (A^TA) = \begin{bmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{bmatrix} \Rightarrow Adj\ (A^TA) = \begin{bmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{bmatrix}$$

STEP 7: Inverse matrix (A^TA)

$$(A^TA)^{-1} = \frac{1}{\det(A^TA)}\ (\ cof\ adj\ (A^TA))$$

$$= \frac{1}{21} \times \begin{bmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{bmatrix} = \begin{bmatrix} \frac{8}{21} & \frac{3}{21} & \frac{1}{21} \\ \frac{3}{21} & \frac{9}{21} & \frac{3}{21} \\ \frac{1}{21} & \frac{3}{21} & \frac{8}{21} \end{bmatrix}$$

STEP 8: Find matrix (A^TL)

$$A^TL = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 16.576 \\ 14.352 \\ -2.455 \\ 14.350 \\ 11.898 \\ 16.563 \\ 4.664 \end{bmatrix} = \begin{bmatrix} 37.803 \\ 4.779 \\ 31.157 \end{bmatrix}$$

STEP 9: Solve $x = (A^TA)^{-1} . A^TL$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{8}{21} & \frac{3}{21} & \frac{1}{21} \\ \frac{3}{21} & \frac{9}{21} & \frac{3}{21} \\ \frac{1}{21} & \frac{3}{21} & \frac{8}{21} \end{bmatrix} \times \begin{bmatrix} 37.803 \\ 4.779 \\ 31.157 \end{bmatrix} = \begin{bmatrix} 16.568 \\ 11.900 \\ 14.352 \end{bmatrix}$$

A = 16.568 m

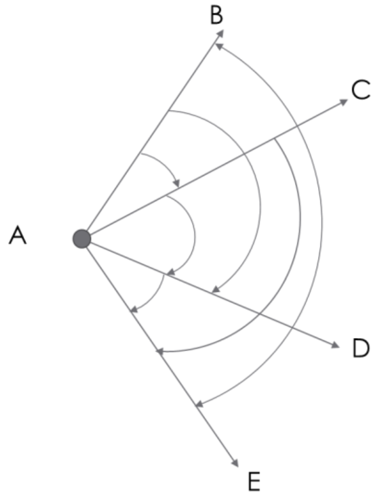
B = 11.900 m

C = 14.352 m

Example 6:

Calculate angle BAC, CAD and DAE using Least Square Adjustment observation equation method.

Position	Angle
BAC	30° 38′56"
CAD	54° 25′20"
DAE	25° 18′40"
BAD	85° 04′24"
CAE	79° 43′55"
BAE	110° 22′50"



Step 1: Model the observation equation

$BAC = 30° 38^{'} 56" + V_1$

$CAD = 54° 25^{'} 20" + V_2$

$DAE = 25° 18^{'} 40" + V_3$

$BAC + \mathrm{CAD} = 85° 04^{'} 24" + V_4$

$CAD + DAE = 79° 43^{'} 55" + V_5$

$BAC + \mathrm{CAD} + \mathrm{DAE} = 110° 22^{'} 50" + V_6$

STEP 2: Create matrix A, X and L

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \mathrm{X} = \begin{bmatrix} BAC \\ CAD \\ DAE \end{bmatrix}; \mathrm{L} = \begin{bmatrix} 30° 38^{'} 56" \\ 54° 25^{'} 20" \\ 25° 18^{'} 40" \\ 85° 04^{'} 24" \\ 79° 43^{'} 55" \\ 110° 22^{'} 50" \end{bmatrix}$$

STEP 3: Find metric A^TA

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^TA}$)

$$\text{Det } A^TA = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = 3 \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} - 2 \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} + 1 \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 16$$

STEP 5: Minor matrix for (A^TA)

$$A^TA = \begin{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

STEP 6: Adjoint matrix A^TA

$$Adj(A^TA) = (cof\ A^TA)^T$$

$$cofactor\ (A^TA) = minor(A^TA) \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Cof\ (A^TA) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \quad \Longrightarrow \quad Adj\ (A^TA) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

STEP 7: Inverse matrix (A^TA)

$(A^TA)^{-1} = \frac{1}{\det(A^TA)}\ (\,adj\ (A^TA))$

$$= \frac{1}{16} \times \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix}$$

STEP 8: Find matrix (A^TL)

$$A^TL = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 30^\circ\ 38'56" \\ 54^\circ\ 25'\ 20" \\ 25^\circ\ 18'40" \\ 85^\circ\ 04'24" \\ 79^\circ\ 43'\ 55" \\ 110^\circ\ 22'50" \end{bmatrix} = \begin{bmatrix} 226^\circ\ 06'10" \\ 329^\circ\ 36'29" \\ 215^\circ\ 25'25" \end{bmatrix}$$

STEP 9: Solve $x = (A^TA\,)^{-1}\,.\,A^TL$

$x = (A^TA\,)^{-1}\,.\,A^TL$

$$\begin{bmatrix} BAC \\ CAD \\ DAE \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.25 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix} \times \begin{bmatrix} 226^\circ\ 06'10" \\ 329^\circ\ 36'29" \\ 215^\circ\ 25'25" \end{bmatrix} = \begin{bmatrix} 30^\circ\ 38'57.75" \\ 54^\circ\ 25'20.75" \\ 25^\circ\ 18'35.25" \end{bmatrix}$$

BAC = 30° 38′57.75"

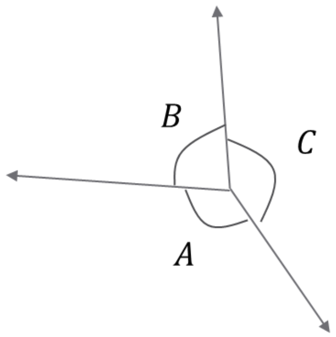
CAD = 54° 25′20.75"

DAE = 25° 18′35.25"

Example7: Condition adjustment

The three observations are related to their adjusted values and their residuals. Calculate adjusted angle for A and B using Least Square Adjustment observation equation method.

Point	Angle
A	$150^\circ\,20'30''$
B	$80^\circ 17'35''$
C	$129^\circ\,21'30''$



Step 1: Model the observation equation

$A\ =\ 150^\circ\,20'30'' +\ V_1$

$B\ =\ 80^\circ 17'35'' +\ V_2$

$C = 129^\circ\,21'30'' +\ V_3$

Condition equation

$A + \mathrm{B} + \mathrm{C} =\ 360^\circ\,00'00''$

$\mathrm{C} =\ 360^\circ\,00'00''\ -\ A - B$

Substitute to 1st equation

$\mathbf{360^\circ\,00'00''\ -\ \mathit{A} - \mathit{B}} = 129^\circ\,21'30'' +\ V_3$

$A + B\ = 360^\circ\ \ - 129^\circ\,21'30'' +\ V_3$

$A + B\ = 230^\circ\,38'30'' +\ V_3$

NEW EQUATION

$A\ =\ 150^\circ\,20'30'' +\ V_1$

$B\ =\ 80^\circ 17'35'' +\ V_2$

$A + B\ = 230^\circ\,38'30'' +\ V_3$

STEP 2: Create matrix A, X and L

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} ; X = \begin{bmatrix} A \\ B \end{bmatrix} ; L = \begin{bmatrix} 150° \ 20' 30" \\ 80° 17' 35" \\ 230° \ 38' 30" \end{bmatrix}$$

STEP 3: Find metric $\boldsymbol{A^T A}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^T A}$)

$$\text{Det } A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (2 \times 2) - (1 \times 1) = 3$$

STEP 5: Minor matrix r for $(\boldsymbol{A^T A})$

$$\text{Cof } A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

STEP 6: Adjoint matrix $\boldsymbol{A^T A}$

$$Adj(A^T A) = \ (cof \ A^T A)^T$$

$$cofactor \ (A^T A) = minor(A^T A) \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$cof \ (A^T A) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Adj \ (A^T A) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

STEP 7: Inverse matrix ($A^T A$)

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \; (\, adj \; (A^T A))$$

$$= \frac{1}{3} \times \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

STEP 8: Find matrix $(\boldsymbol{A^T L})$

$$A^T L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 150° \, 20^{'} 30" \\ 80° 17^{'} 35" \\ 230° \, 38^{'} 30" \end{bmatrix} = \begin{bmatrix} 380° \, 59^{'} 00" \\ 310° \, 56^{'} 05" \end{bmatrix}$$

STEP 9: Solve $\boldsymbol{x = (A^T A\,)^{-1} \, . A^T L}$

$$x = (A^T A\,)^{-1} \, . A^T L$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} 380° \, 59^{'} 00" \\ 310° \, 56^{'} 05" \end{bmatrix} = \begin{bmatrix} 150° \, 20^{'} 38.33" \\ 80° \, 17^{'} 43.33" \end{bmatrix}$$

A $= 150° \; 20' 38.33"$

B $= 80° \; 17' 43.33"$

C $= \; 360° - (\, 150° \; 20' 38.33" + \; 80° \; 17' 43.33" \,)$

$= 129° \; 21' 38.34"$

SOLVE LEAST SQUARE ADJUSTMENT WITH WEIGHTS

Step by step to solve LSA problem with weights

10. Model the observation equation
11. Create matrix A, X, W and L
12. Find matrix $A^T WA$
13. Find Determinant for $A^T WA$
14. Find minor matrix for $A^T WA$
15. Adjoint matrix $A^T WA$
16. Inverse matrix $A^T WA$
17. Find $A^T WL$
18. Solve $x = (A^T WA)^{-1} . A^T WL$

Example 1

Calculate the variables for the elevations of A, B and C. Use the least square adjustment method. Given the elevation of BM 1 is 15.384m and BM2=16.245m

From	To	Different Height	Weights
A	BM1	-5.663	2
BM 1	C	-2.929	2
C	B	5.174	2
C	BM2	3.790	4
BM2	B	1.378	1
BM2	A	4.802	2
B	A	3.420	4

Step 1: Model the observation equation

$BM1 - A = -5.663 + V_1$

$C - BM1 = -2.929 + V_2$

$B - C = 5.174 + V_3$

$BM2 - C = 3.790 + V_4$

$B - BM2 = 1.378 + V_5$

$A - BM2 = 4.802 + V_6$

$A - B = 3.420 + V_7$

NOTE :

- Different height = fore sight - back sight
- Insert know value into the equation

New equation after insert know value

$\mathbf{15.384} - A = -5.663 + V_1$

$C - \mathbf{15.384} = -2.929 + V_2$

$B - C = 5.174 + V_3$

$\mathbf{16.245} - C = 3.790 + V_4$

$B - \mathbf{16.245} = 1.378 + V_5$

$A - \mathbf{16.245} = 4.802 + V_6$

$A - B = 3.420 + V_7$

$A = 21.047 + V_1$

$C = 12.455 + V_2$

$B - C = 5.174 + V_3$

$C = 12.455 + V_4$

$B = 17.623 + V_5$

$A = 21.047 + V_6$

$A - B = 3.420 + V_7$

STEP 2: Create matrix A, X, W and L

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad X = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad L = \begin{bmatrix} 21.047 \\ 12.455 \\ 5.174 \\ 12.455 \\ 17.623 \\ 21.047 \\ 3.420 \end{bmatrix} \quad W = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Base on weighted data

STEP 3: Find metric A^TWA

$$A^TA = \left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}\right] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 0 & 1 & 0 & -4 \\ 0 & 2 & -2 & 4 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 7 & -2 \\ 0 & -2 & 8 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathrm{A^TW\,A}$)

$$\text{Det } A^TA = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 7 & -2 \\ 0 & -2 & 8 \end{bmatrix} = 8\begin{bmatrix} 7 & -2 \\ -2 & 8 \end{bmatrix} - (-4)\begin{bmatrix} -4 & -2 \\ 0 & 8 \end{bmatrix} + 0\begin{bmatrix} -4 & 7 \\ 0 & -2 \end{bmatrix} = 288$$

STEP 5: Minor matrix for (A^TWA)

$$\text{Minor } A^TA = \begin{bmatrix} \begin{bmatrix} 7 & -2 \\ -2 & 8 \end{bmatrix} & \begin{bmatrix} -4 & -2 \\ 0 & 8 \end{bmatrix} & \begin{bmatrix} -4 & 7 \\ 0 & -2 \end{bmatrix} \\ \begin{bmatrix} -4 & 0 \\ -2 & 8 \end{bmatrix} & \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} & \begin{bmatrix} 8 & -4 \\ 0 & -2 \end{bmatrix} \\ \begin{bmatrix} -4 & 0 \\ 7 & -2 \end{bmatrix} & \begin{bmatrix} 8 & 0 \\ -4 & -2 \end{bmatrix} & \begin{bmatrix} 8 & -4 \\ -4 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 52 & -32 & 8 \\ -32 & 64 & -16 \\ 8 & -16 & 40 \end{bmatrix}$$

STEP 6: Adjoint matrix A^TWA

$$cofactor\ (A^TWA) = \begin{bmatrix} 52 & 32 & 8 \\ 32 & 64 & 16 \\ 8 & 16 & 40 \end{bmatrix} \quad \Longrightarrow \quad Adj\ (A^TWA) = \begin{bmatrix} 52 & 32 & 8 \\ 32 & 64 & 16 \\ 8 & 16 & 40 \end{bmatrix}$$

STEP 7: Inverse matrix (A^TWA)

$$(A^TWA)^{-1} = \frac{1}{\det(A^TWA)}\ (\,cof\ adj\ (A^TWA))$$

$$= \frac{1}{288} \times \begin{bmatrix} 52 & 32 & 8 \\ 32 & 64 & 16 \\ 8 & 16 & 40 \end{bmatrix} \quad = \begin{bmatrix} \frac{13}{72} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{18} \\ \frac{1}{36} & \frac{1}{18} & \frac{5}{36} \end{bmatrix}$$

STEP 8: Find matrix (A^TWL)

$$A^TWL = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \times \begin{bmatrix} 21.047 \\ 12.455 \\ 5.174 \\ 12.455 \\ 17.623 \\ 21.047 \\ 3.420 \end{bmatrix}$$

$$A^TWA = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 0 & 1 & 0 & -4 \\ 0 & 2 & -2 & 4 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 21.047 \\ 12.455 \\ 5.174 \\ 12.455 \\ 17.623 \\ 21.047 \\ 3.420 \end{bmatrix} = \begin{bmatrix} 97.868 \\ 14.291 \\ 64.382 \end{bmatrix}$$

STEP 9: Solve $x = (A^TWA\,)^{-1}\,.\,A^TWL$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{13}{72} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{18} \\ \frac{1}{36} & \frac{1}{18} & \frac{5}{36} \end{bmatrix} \times \begin{bmatrix} 97.868 \\ 14.291 \\ 64.382 \end{bmatrix} = \begin{bmatrix} 21.103 \\ 17.641 \\ 12.490 \end{bmatrix}$$

A = 21.103 m

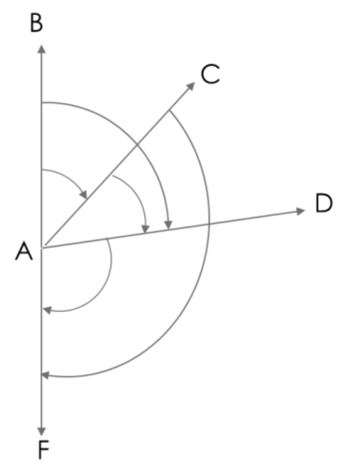
B = 17.641 m

C = 12.490 m

Example 2:

Calculate angle BAC, CAD and DAF using Least Square Adjustment observation equation method.

Position	Angle	Std. dev
BAC	45° 38′56"	5"
CAD	48° 25′20"	2"
DAF	85° 55′45"	2"
BAD	94° 04′20"	5"
CAF	134° 21′05"	10"



Step 1: Model the observation equation

$$BAC = 45^\circ\, 38'\, 56'' + V_1$$

$$CAD = 48^\circ\, 25'\, 20'' + V_2$$

$$DAF = 85^\circ\, 55'\, 45'' + V_3$$

$$BAC + CAD = 94^\circ\, 04'\, 20'' + V_4$$

$$\mathrm{CAD} + \mathrm{DAF} = 134^\circ\, 21'\, 05'' + V_5$$

Condition equation

$$\mathrm{BAC} + \mathrm{CAD} + \mathrm{DAF} = 180^\circ$$

$$\mathrm{DAF} = 180^\circ - (\mathrm{BAC} + \mathrm{CAD})$$

Substitute into observation equation

$$BAC = 45^\circ\, 38'\, 56'' + V_1$$

$$CAD = 48^\circ\, 25'\, 20'' + V_2$$

$$\mathbf{180^\circ - BAC - CAD} = 85^\circ\, 55'\, 45'' + V_3$$

$$BAC + CAD = 94^\circ\, 04'\, 20'' + V_4$$

$$\mathrm{CAD} + (\mathbf{180^\circ - BAC - CAD}) = 134^\circ\, 21'\, 05'' + V_5$$

New Equation

$$BAC = 45^\circ\, 38'\, 56'' + V_1$$

$$CAD = 48^\circ\, 25'\, 20'' + V_2$$

$$\mathbf{BAC + CAD} = 94^\circ\, 04'\, 15'' + V_3$$

$$BAC + CAD = 94^\circ\, 04'\, 20'' + V_4$$

$$\mathbf{BAC} = 45^\circ\, 38'\, 55'' + V_5$$

STEP 2: Create matrix A, X ,W and L

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}; X = \begin{bmatrix} BAC \\ CAD \end{bmatrix}; \quad L = \begin{bmatrix} 45° \ 38'56" \\ 48° \ 25' \ 20" \\ 94° \ 04'15" \\ 94° \ 04'20" \\ 45° \ 38' \ 55" \end{bmatrix} \quad W = \begin{bmatrix} \frac{1}{5^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{10^2} \end{bmatrix}$$

STEP 3: Find metric A^T W A

$$A^TWA = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{5^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{10^2} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^TWA = \begin{bmatrix} 0.04 & 0 & 0.25 & 0.04 & 0.01 \\ 0 & 0.25 & 0.25 & 0.04 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.34 & 0.29 \\ 0.29 & 0.54 \end{bmatrix}$$

STEP 4: Find Determinant for matrix ($\mathbf{A^TW\,A}$)

Det $A^TWA = \begin{bmatrix} 0.34 & 0.29 \\ 0.29 & 0.54 \end{bmatrix} = 0.0995$

STEP 5: Minor matrix r for $(\boldsymbol{A^TWA})$

minor $A^TWA = \begin{bmatrix} 0.54 & 0.29 \\ 0.29 & 0.34 \end{bmatrix}$

STEP 6: Adjoint matrix A^TWA

$Adj(A^TWA) = (cof\ A^TWA)^T$

$$cofactor\ (A^TWA) = minor(A^TWA)\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$cof\ (A^TWA) = \begin{bmatrix} 0.54 & 0.29 \\ 0.29 & 0.34 \end{bmatrix} \Rightarrow Adj\ (A^TWA) = \begin{bmatrix} 0.54 & -0.29 \\ -0.29 & 0.34 \end{bmatrix}$$

STEP 7: Inverse matrix (A^TWA)

$$(A^TWA)^{-1} = \frac{1}{\det(A^TWA)}\ (adj\ (A^TWA))$$

$$= \frac{1}{0.0995} \times \begin{bmatrix} 0.54 & -0.29 \\ -0.29 & 0.34 \end{bmatrix} = \begin{bmatrix} \frac{1080}{199} & \frac{-580}{199} \\ \frac{-580}{199} & \frac{680}{199} \end{bmatrix}$$

STEP 8: Find matrix (A^TWL)

$$A^TWL = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{10^2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 45°\,38'\,56" \\ 48°\,25'\,20" \\ 94°\,04'\,15" \\ 94°\,04'\,20" \\ 45°\,38'\,55" \end{bmatrix}$$

$$= \begin{bmatrix} 0.04 & 0 & 0.25 & 0.04 & 0.01 \\ 0 & 0.25 & 0.25 & 0.04 & 0 \end{bmatrix} \cdot \begin{bmatrix} 45°\,38'\,56" \\ 48°\,25'\,20" \\ 94°\,04'\,15" \\ 94°\,04'\,20" \\ 45°\,38'\,55" \end{bmatrix} = \begin{bmatrix} 29°33'47" \\ 39°23'10" \end{bmatrix}$$

STEP 9: Solve $\boldsymbol{x} = (\boldsymbol{A^TWA})^{-1} . \boldsymbol{A^TWL}$

$$x = (A^TWA)^{-1} . A^TWL$$

$$\begin{bmatrix} BAC \\ CAD \end{bmatrix} = \begin{bmatrix} \frac{1080}{199} & \frac{-580}{199} \\ \frac{-580}{199} & \frac{680}{199} \end{bmatrix} \times \begin{bmatrix} 29°33'47" \\ 39°23'10" \end{bmatrix} = \begin{bmatrix} 45°\ 38'58.48" \\ 48°\ 25'19.3" \end{bmatrix}$$

BAC $= 45°\ 38'58.48"$

CAD $= 48°\ 25'19.3"$

DAF $= 180° - (\ 45°\ 38'58.48" + \ 48°\ 25'19.3"\ )$

$= 85°\ 55'42.22"$

> **"You don't have to be great to start, but you have to start to be great"**
>
> **-Zig Ziglar-**

Tutorial

Question 1

Calculate the adjustment length AD and its estimated error given Figure 3 and the observation data below

line	Distance, m
AB	3.17
BC	1.12
CD	2.25
AC	4.31
AD	6.51
BD	3.36

Question 2

The use of least square adjustment principle is to solve the redundant equations. From the equations below:

$$2x + y = 21 + V_1$$
$$24x - 6y = 11 + V_2$$
$$4x - 2y = 20 + V_3$$

i. State the number of variables and observation
ii. Calculate the variable by using the principles of least square adjustment.

Question 3

Using the conditional equation method, what are the most probable values for the three interior angles of a triangle that were measured as.

station	angle	Std. dev
A	$58° 14' 56''$	5.2"
B	$65° 03' 34''$	5.2"
C	$56° 40' 20''$	5.2"

Question 4

Calculate the variables for the elevations of B, C and D. Use the least square adjustment method. Given the elevation of BM 1 is 40.213m

From	To	elevation	Std. dev
A	B	10.509	0.006
B	C	5.360	0.004
C	D	-8.523	0.005
D	A	-7.348	0.003
B	D	-3.167	0.004
A	C	15.881	0.012

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SURVEY ADJUSTMENT provides the students with knowledge on adjustment. The book emphasizes the calculation of adjustment using the least square adjustment method through observation and condition equations in solving surveyed data. Besides, it is also provides students with knowledge and practical skills to calculate and adjust surveyed data.

"**True value** of measurement is **unknown**
Actual size of error is unknown
Errors exist in **measurement data** & **computed results**"

e ISBN 978-967-2241-83-6

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